Large-N expansion at the unitarity point

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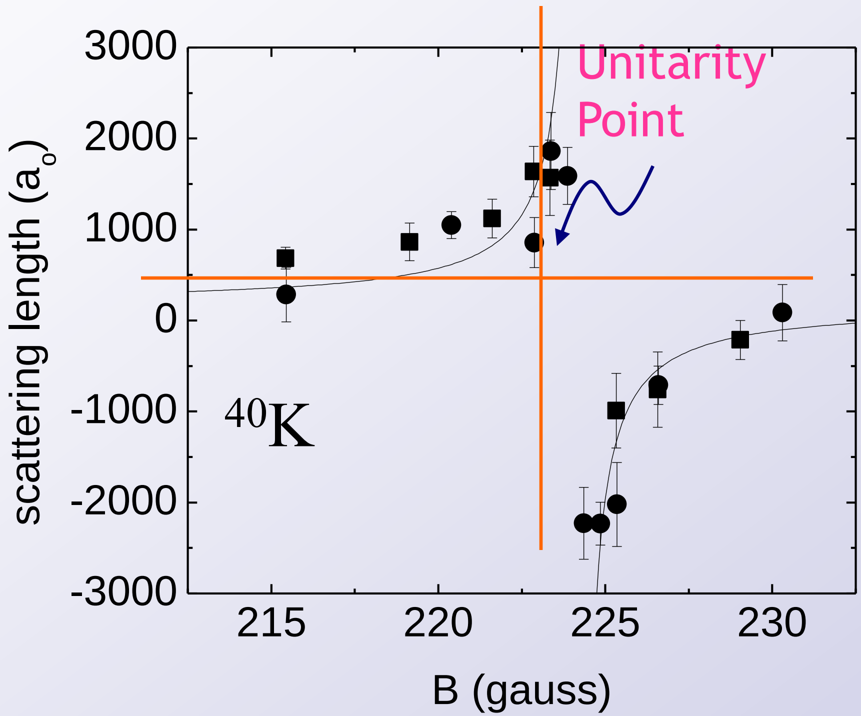
Outline

- Problem and Motivation:
  - Universal scaling at Unitarity point
  - Failure of Low density expansion
- Large N formulation
  - Model and formulation
  - 1/N expansion: Saddle Point + Fluctuations
- Results
  - At critical temperature, Zero Temperature, Finite Polarization
  - Away from unitarity
  - Excitation gap vs. Order parameter

Conclusion

Feshbach Resonance:

$$\mathcal{H} = \sum_{\sigma=\uparrow,\downarrow} \int d^3\mathbf{r}\, \psi_\sigma^\dagger(\mathbf{r}) \left(-\frac{\hbar^2 \nabla_\mathbf{r}^2}{2m} - \mu_\sigma \right) \psi_\sigma(\mathbf{r}) + \lambda \int d^3\mathbf{r}\, \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \psi_\uparrow(\mathbf{r}),$$



C.A. Regal and D.S. Jin, PRL, (2003)

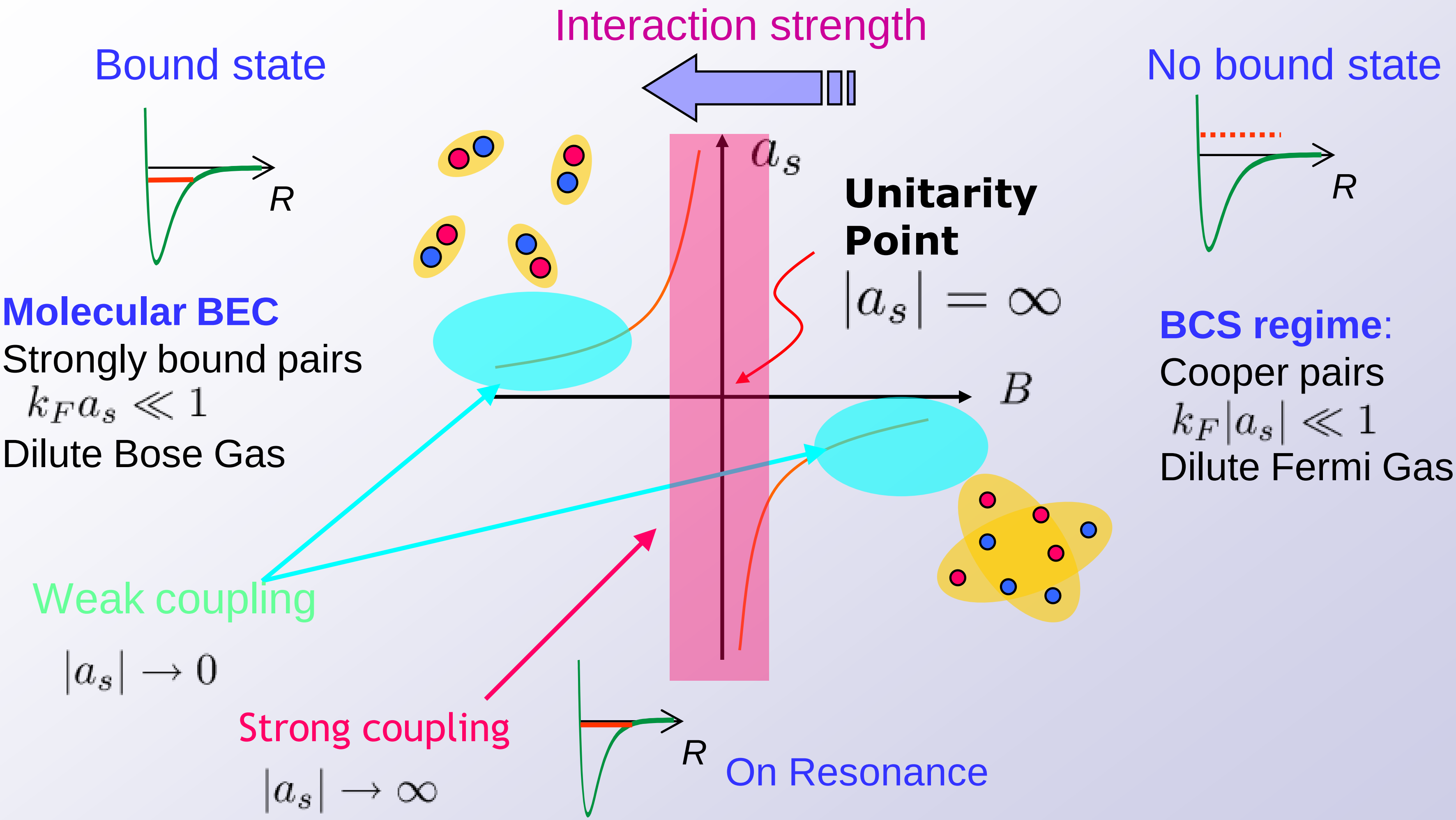
$$\frac{m}{4\pi\hbar^2 a_s} = \frac{1}{\lambda} + \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2\epsilon_\mathbf{k}},$$

$$a(B) = a_{\rm bg} \left[1 - \frac{w}{B - B_{\rm res}} \right]$$

$$F = \epsilon_F \times g\left(k_{\rm F} a_s, \frac{k_B T}{\epsilon_F} \right)$$

$$k_F = (3\pi^2 n)^{1/3}$$

BCS-BEC Crossover



At unitarity: No small parameter to expand around: hard problem

Bertsch: Challenge problem in many-body physics (1998): ground state of resonant gas

Unitarity Limit ($|a_s| = \infty$,Strong Coupling)

k_F is the only scale in the problem !

Simple system, with simple scaling → Energy per particle

$$\varepsilon = \xi \times \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}$$

ξ is a universal parameter, related to many quantities

Chemical Potential $\mu = \xi \times \frac{\hbar^2 k_F^2}{2m}$

Bulk Modulus $B = \xi \times \frac{2}{3}\frac{k_F^3}{3\pi^2}\frac{\hbar^2 k_F^2}{2m}$

Pressure density $p = \xi \times \frac{2}{5}\frac{\hbar^2 k_F^2}{2m}$

First Sound $v = \sqrt{\xi} \times \sqrt{2/3}\frac{\hbar k_F}{m}$

Hard problem: No expansion parameter

Borrow from Critical Phenomena Technology:

Large-N expansion

$$\mathcal{H}_N = \sum_{i=1}^{N} \sum_{\sigma=\uparrow,\downarrow} \int d^3\mathbf{r}\, \psi^\dagger_{i\sigma}(\mathbf{r}) \left(-\frac{\nabla^2_{\mathbf{r}}}{2m} - \mu \right) \psi_{i\sigma}(\mathbf{r})$$
$$+ \frac{\lambda}{N} \sum_{i,j=1}^{N} \int d^3\mathbf{r}\, \psi^\dagger_{i\uparrow}(\mathbf{r}) \psi^\dagger_{i\downarrow}(\mathbf{r}) \psi_{j\downarrow}(\mathbf{r}) \psi_{j\uparrow}(\mathbf{r}),$$

$\mathcal{H} \sim N$, In the limit $N = \infty$,the free energy is given by saddle point contribution

$$F = -k_B T \ln \mathrm{Tr}\left[e^{-\mathcal{H}_N / k_B T} \right]$$

Systematic expansions for ξ and other observables (Δ, Tc, …) in terms of 1/N

Saddle Point Approximation

$$f^{(0)} = N(-\frac{m|\Delta|^2}{4\pi a_s} - \frac{1}{V}\sum_{\mathbf{k}}\left(E_{\mathbf{k}} - \xi_{\mathbf{k}} - \frac{|\Delta|^2}{2\epsilon_{\mathbf{k}}}\right) - \frac{2}{\beta V}\sum_{\mathbf{k}} \ln\left[1 + e^{-\beta E_{\mathbf{k}}}\right]),$$

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta|^2}$$

Gap equation

$$\frac{m\Delta}{4\pi a_s} = \frac{\Delta}{V}\sum_{\mathbf{k}}\left(\frac{\tanh(\beta E_{\mathbf{k}}/2)}{2E_{\mathbf{k}}} - \frac{1}{2\epsilon_{\mathbf{k}}}\right).$$

$$\Delta_o^{(0)}/\epsilon_F = 0.6864,$$
$$\mu_o^{(0)}/\epsilon_F = 0.5906,$$
$$k_B T_c^{(0)}/\epsilon_F = 0.4965,$$
$$\mu_c^{(0)}/\epsilon_F = 0.7469.$$

⇒ $N = \infty$

Recover Standard Mean-Field Results

The $N = \infty$ phase is in the same universality class as $N = 1$

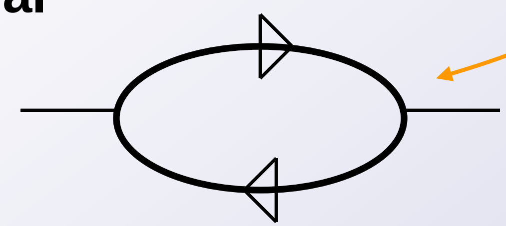
1/N expansion (Loop expansion)

Gaussian Fluctuations around saddle point

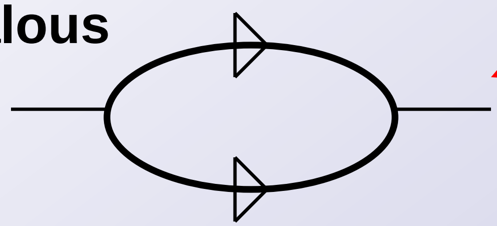
$$f = f_{(0)} + \frac{1}{N} f_{1/N}$$

$$S^{(1/N)} = \frac{1}{2}\sum_q \begin{pmatrix} \hat{b}^*(q) & \hat{b}(-q) \end{pmatrix} \begin{pmatrix} A(q) & B(q) \\ B^*(q) & A(-q) \end{pmatrix} \begin{pmatrix} \hat{b}(q) \\ \hat{b}^*(-q) \end{pmatrix},$$

Normal



Anomalous



$$\Delta = \Delta_o^{(0)} + \frac{1}{N}\delta\Delta_o + \ldots,$$

$$\mu = \mu_o^{(0)} + \frac{1}{N}\delta\mu_o + \ldots$$

$$T_c = T_c^{(0)} + \frac{1}{N}\delta T_c + \ldots$$

$$\mu = \mu_c^{(0)} + \frac{1}{N}\delta\mu_c + \ldots$$

Result

Zero Temperature

$$\Delta_o/\epsilon_F = 0.6864 - \frac{0.163}{N} + \ldots,$$

$$\mu_o/\epsilon_F = 0.5906 - \frac{0.312}{N} + \ldots$$

Result from 1/N $\xi = 0.5906 - 0.312/N + \cdots$

Exp with ^{40}K $\xi = 0.46^{+0.05}_{-0.12}$

Critical Temperature

$$T_c/\epsilon_F = 0.4964 - \frac{1.31}{N} + \ldots$$

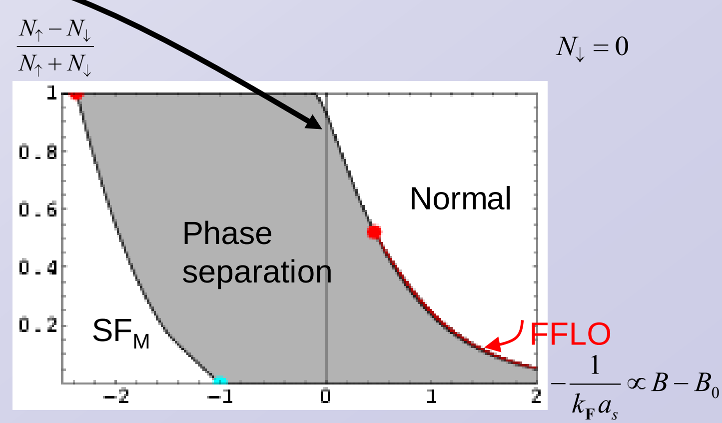
$$\mu_c/\epsilon_F = 0.7469 - \frac{0.58}{N} + \ldots$$

Population Imbalance

$$h_{c2}/\epsilon_F = 0.6929 + \frac{0.087}{N} + \ldots,$$

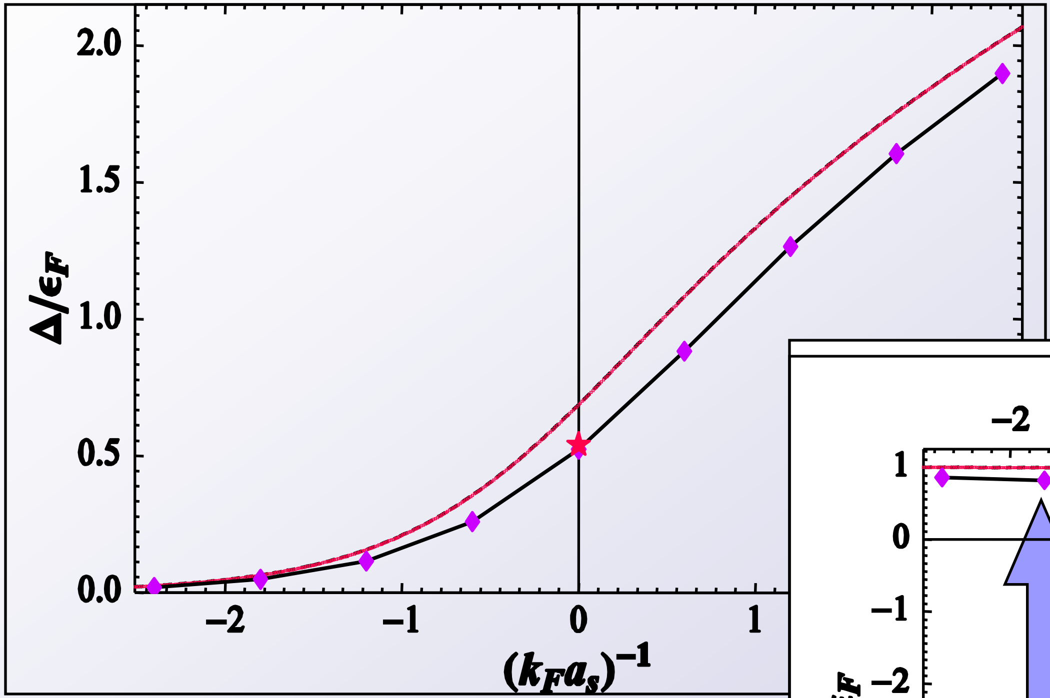
$$P_{c2} = 0.9326 - \frac{0.631}{N} + \ldots$$

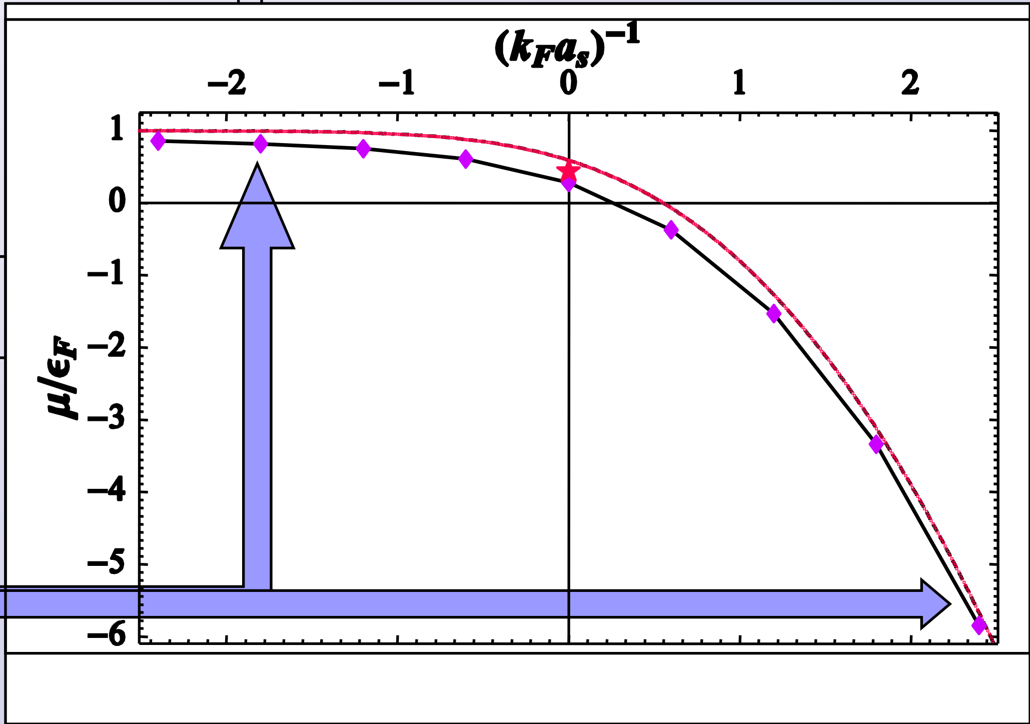
$$\mu/\epsilon_F = 0.8585 - \frac{0.458}{N} + \ldots$$



BEC limit **BCS limit**

Large-N calculation away from Unitarity



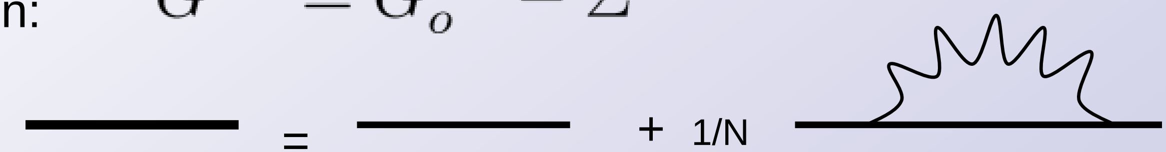


Recover Standard Result Away from Unitarity

Excitation Gap Vs Order Parameter

- Excitation gap: Δ_{exc} Minimum energy to create excitation
- Order Parameter: $\Delta = \lambda\langle\psi^\dagger\psi^\dagger\rangle$
- In general $\Delta_{exc} \neq \Delta$ (but equal in Mean-field theory)

Δ_{exc} Calculation: $$G^{-1} = G_o^{-1} - \Sigma$$



$$\Delta_{exc} = \Delta + \frac{1}{2}\left(\Sigma_{11} + \Sigma_{22} - 2\Sigma_{12}\right)\Big|_{|\mathbf{k}|=\sqrt{2m\mu_o^{(0)}},\; i\omega=\Delta_o^{(0)}}$$

$$\Delta_{exc}/\epsilon_F = 0.6864 - 0.196/N + \mathcal{O}(1/N^2),$$

$$\Delta_o/\epsilon_F = 0.6864 - 0.163/N + \mathcal{O}(1/N^2),$$

Monte Carlo $\Delta_{mc}/\epsilon_F = 0.54$ (Carlson et al, 2003)

Parallel effort

- Large N,
  - Nikolic and Sachdev, cond-mat/0609106 (also talk in this session, **L32.00011**)
- ε expansion (4- ε and 2+ε)
  - Nussinov and Nussinov, PRA (2006)
  - Nishida and Son, PRL (2006) and cond-mat/0607835; Nishida, cond-mat/0608321
  - Arnold, Drut and Son, cond-mat/0608477

Summary

- Investigate universal properties near unitarity point using systematic 1/N expansion
  - $N = \infty$ is exact solution
- Derive the 1/N contributions to various physical quantities
- Future: Dynamic and Spectral function.
  - Make contact with recent experiment from Ketterle's group on normal state at large polarization

Cold Atoms
Basic Relations

	E [eV]	v [cm s^{-1}]	λ [Å]	h [cm]
E [eV]		$5.182\ 10^{-13}\ A\ v^2$	$\frac{0.0825}{A\ \lambda^2}$	$1.017\ 10^{-9}\ A\ h$
v [cm s^{-1}]	$1.389\ 10^6 \sqrt{\frac{E}{A}}$		$\frac{3.990\ 10^5}{A\ \lambda}$	$44.29 \sqrt{h}$
λ [Å]	$\frac{0.2873}{\sqrt{E\ A}}$	$\frac{3.990\ 10^5}{A\ v}$		$\frac{9008.6}{A\ \sqrt{h}}$
h [cm]	$9.836\ 10^8 \frac{E}{A}$	$5.097\ 10^{-4}\ v^2$	$\frac{8.115\ 10^7}{A^2\ \lambda^2}$	

$$t = 0.04515 \sqrt{h}$$

Cold Atoms
Basic Relations

TABLE I. – *Energy and length scales for trapped gaseous Bose-Einstein condensates. The hierarchy of energy and length scales simplifies the description of these quantum fluids. For each energy E, we define a length l by the relation $E = \hbar^2/2ml^2$ and indicate the relation between l and a common length scale. Numbers are typical for sodium BEC experiments. The values for the mean-field energy assume a density of $\sim 10^{14}$ cm^{-3}.*

Energy Scale E	$= \hbar^2/2ml^2$	Length Scale		
limiting temperature for s-wave scattering	1 mK	scattering length	$a = l/2\pi$	= 3 nm
BEC transition temperature T_c	2 μK	separation between atoms	$n^{-1/3} = l/\sqrt{\pi}(2.612)^{1/3}$	= 200 nm
single-photon recoil energy	1.2 μK	optical wavelength	$\lambda = l$	= 600 nm
temperature T	1 μK	thermal de Broglie wavelength	$\lambda_{dB} = l/\sqrt{\pi}$	= 300 nm
mean field energy μ	300 nK	healing length	$\xi = l/2\pi$	= 200 nm
harmonic oscillator level spacing $\hbar\omega$	0.5 nK	oscillator length ($\omega \simeq 2\pi \cdot 10$Hz)	$a_{HO} = l/\sqrt{2}\pi$	= 6.5 μm

Large-N calculation

- Problem can be solved exactly for N=∞
  - N=∞ solution is also superfluid (Good)

But interested in N=1 problem….

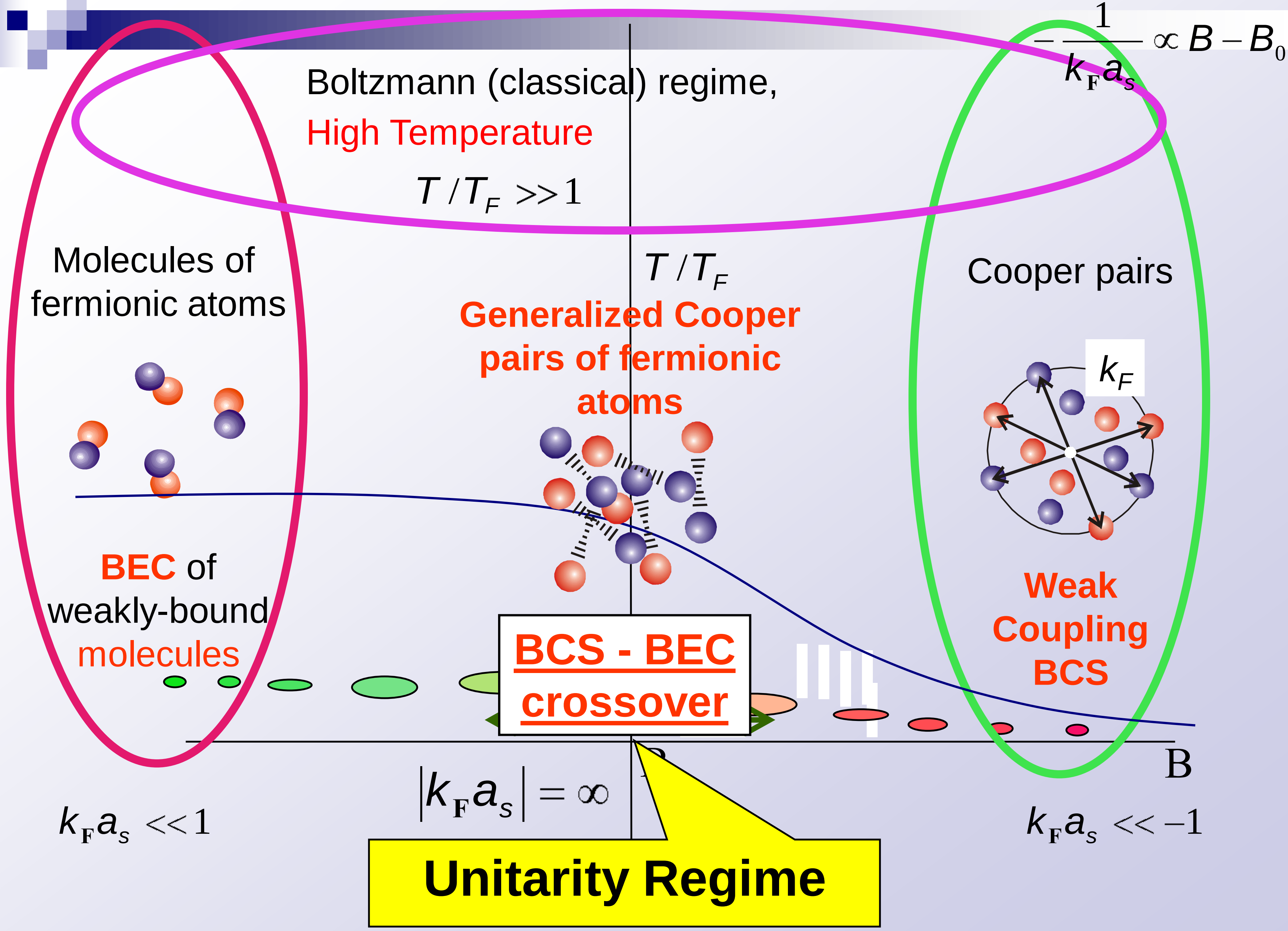
  - Perform 1/N expansion

Given by Feynman diagrams

$$F = F_\infty + \frac{1}{N} F'_\infty + \frac{1}{N^2}\frac{1}{2} F''_\infty + \cdots ,$$

  - Very successful approach for critical phenomena and high energy physics

At unitarity:	N=∞	1/N expansion at N=1	Monte Carlo
ϵ_F/T_c	2.01	7.32	6.59
Δ/ϵ_F	0.69	0.52	0.54



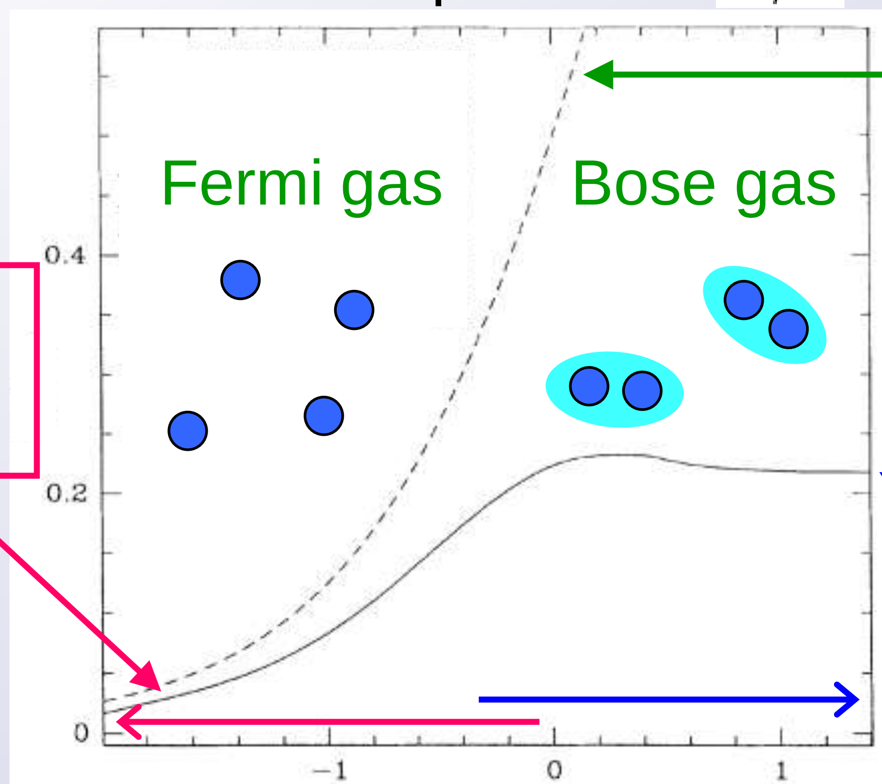
$-\frac{1}{k_F a_s} \propto B - B_0$
Boltzmann (classical) regime,
High Temperature
$T/T_F >> 1$
Molecules of fermionic atoms
T/T_F
Generalized Cooper pairs of fermionic atoms
Cooper pairs
k_F
BEC of weakly-bound molecules
Weak Coupling BCS
BCS - BEC crossover
B
$k_F a_s << 1$
$|k_F a_s| = \infty$
$k_F a_s << -1$
Unitarity Regime

BCS-BEC Crossover at finite T

Sa de Melo et al., PRL 71 (1993) 3202

$$\mathcal{H} = \bar{\psi}_\sigma(x)\left[-\frac{\nabla^2}{2m} - \mu\right]\psi_\sigma(x) - g\bar{\psi}_\uparrow(x)\bar{\psi}_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x)$$

critical temperature: T_c/ϵ_F



scattering length: $1/(k_F a_s)$

Dissociation of bound boson

$E_b/[\log(E_b/\epsilon_F)]^{2/3}$

BCS behavior

$T_c/\epsilon_F \sim e^{-1/k_F|a_s|}$

BEC behavior

$T_c/\epsilon_F \sim \text{const.}$