# Establishing Density Profile of Bose-Einstein Condensate in a Harmonic Potential


## Abstract:

We have successfully managed to establish density profiles for Bose-Einstein condensates in a harmonic potential in both one and three dimensions. This further supports the analytical evaluations and conclusions which were presented by Albert Einstein in the 1920's and the current experiments where BECs are produced today. we have further concluded that it's existence is supported through sheer mathematical evaluation without the use of experiments.


## Equations:

$$V(x) = \frac{1}{2}m\omega^2 x^2 \quad ...(1)$$

$$\psi_i(x) = \sqrt{\frac{1}{2^i}i!}\cdot\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\cdot e^{-\frac{m\omega x^2}{2\hbar}}\cdot H_i\left(\sqrt{\tfrac{m\omega}{\hbar}}x\right), \qquad i=0,1,2,.... \quad ...(2)$$

$$n_i = \frac{1}{e^{(\epsilon_i-\mu)/kT}-1} \quad ...(3)$$

$$\epsilon_i = (i+\frac{1}{2})\hbar\omega \quad ...(4)$$

$$n_{total} = \sum_{i=0}^{\infty}\frac{1}{e^{(\epsilon_i-\mu)/kT}-1} \quad ...(5)$$

$$\Psi_q(z) = -log(1-q) + log(q)\cdot\sum_{i=o}^{\infty}\frac{q^{i+z}}{1-q^{i+z}} \quad ...(6)$$

$$\sum_{i=o}^{\infty}\frac{q^{i+z}}{1-q^{i+z}} = \frac{\Psi_q(z)+log(1-q)}{log(q)} \quad ...(7)$$

$$0 = n_{total}(\tilde{T},\tilde{m}) - \frac{\Psi_{e^{-\frac{1}{T}}}(\frac{1}{2}-\tilde{m}) + log(1-e^{-\frac{1}{T}})}{log(e^{-\frac{1}{T}})} \quad ...(8)$$

$$n_i(j) = \frac{1}{e^{\frac{(i+\frac{1}{2})-\tilde{m}(j)}{\tilde{T}(j)}}-1} \quad ...(9)$$

$$\rho(j,x) = \sum_{i=o}^{\infty} n_i(j)\cdot\left|\Psi_i(x)\right|^2 \quad ...(10)$$

$$\varrho(j,\tilde{x}) = \sum_{i=o}^{\infty} n_i(j)\cdot|\Phi_i(\tilde{x})|^2 = \frac{1}{\sqrt{\pi}}\sum_{i=o}^{\infty}\frac{1}{e^{\frac{(i+\frac{1}{2})-\tilde{m}(j)}{\tilde{T}(j)}}-1}\cdot\frac{1}{2^i i!}(H_i(\tilde{x}))^2 e^{-\tilde{x}^2} \quad ...(11)$$

$$V(r) = \frac{1}{2}\mu\omega^2 r^2 \quad ...(12)$$

$$\Psi_{klm}(r,\theta,\phi) = N_{kl}r^l e^{-vr^2}L_k^{l+\frac{1}{2}}(2vr^2)Y_{lm}(\theta,\phi) \quad ...(13)$$

$$N_n = \frac{(n+1)(n+2)}{2}\frac{1}{e^{\frac{\hbar\omega}{KT}(n+\frac{3}{2})-\frac{\mu}{KT}}-1}, \quad ...(14)$$

$$N_{total} = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}\frac{1}{e^{\frac{\hbar\omega}{KT}(n+\frac{3}{2})-\frac{\mu}{KT}}-1} \quad ...(15)$$

$$N_{total} = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{n^2}{e^{\tilde{T}(n-\tilde{m})}-1}+\frac{3n}{e^{\tilde{T}(n-\tilde{m})}-1}+\frac{2}{e^{\tilde{T}(n-\tilde{m})}-1}\right) \quad ...(16)$$

$$\sum_{n=0}^{\infty}\frac{1}{e^{\tilde{T}(n-\tilde{m})}-1} \approx \sum_{n=0}^{n_l}\frac{1}{e^{\tilde{T}(n-\tilde{m})}-1}+\sum_{n=n_l}^{\infty}\frac{1}{e^{\tilde{T}(n-\tilde{m})}} \quad ...(17)$$

$$D = \sum_{n=n_l}^{\infty}\frac{1}{e^{\tilde{T}(n-\tilde{m})}} = \frac{e^{\tilde{T}(\tilde{m}-n_l)}}{1-e^{-\tilde{T}}} \quad ...(18)$$

$$0 = N_{total} - \sum_{n=0}^{n_l}\frac{n^2}{e^{-\tilde{T}(n-\tilde{m})}-1} + \left(-2\tilde{m}\frac{\partial D}{\partial\tilde{T}}+\frac{\partial^2 D}{\partial\tilde{T}^2}+(\tilde{m})^2 D\right) + 3\left(\sum_{n=0}^{n_l}\frac{n}{e^{-\tilde{T}(n-\tilde{m})}-1}+\tilde{m}D-\frac{\partial D}{\partial\tilde{T}}\right) + 2\left(\sum_{n=0}^{n_l}\frac{1}{e^{-\tilde{T}(n-\tilde{m})}-1}+D\right) \quad ...(19)$$

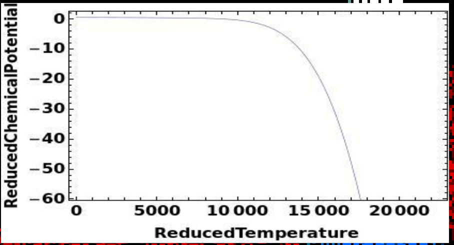
$$\rho(r,\theta,\phi) = \sum_{n=1}^{\infty}\frac{1}{e^{\tilde{T}(n+\frac{1}{2}-\tilde{m})}-1}\left|\Psi_{nlm}(r,\theta,\phi)\right|^2 \quad ...(20)$$

$$\sum_{m=-l}^{l}|Y_{lm}(\theta,\phi)|^2 = \frac{2l+1}{4\pi} \quad ...(21)$$

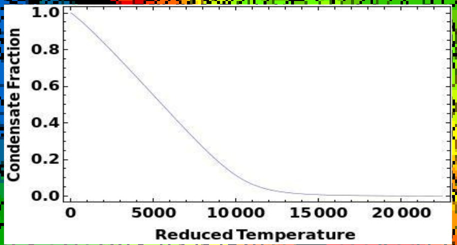
$$\rho(\tilde{r}) = a^3\tilde{p}(\tilde{r}) = \sum_{n=1}^{n_{max}}\frac{1}{e^{\tilde{T}(n+\frac{1}{2}-\tilde{m})}-1}\sum_{l=0}^{n-1}\frac{1}{\pi^{\frac{3}{2}}}\frac{2^{k+l}k!}{(n+l+1)!!}(\tilde{r})^{2l}e^{-(\tilde{r})^2}\left(L_k^{l+\frac{1}{2}}((\tilde{r})^2)\right)^2(2l+1) \quad (22)$$

## Establishing Density Profile for BEC in 1-D Harmonic Potential:

The potential trap of a simple harmonic oscillator is given by (1), and the solution to the Schrodinger equation is given by (2). BEC distribution of the mean occupation of particles is given by (3) and the energy of the $i^{th}$ energy level is given by (4). We know the total number of particles in the system can be expressed by (5). We desire to establish the relationship between the chemical potential (mu) and the temperature (T). Utilizing the Q-Polygamma function (6) we can establish numerical solutions to the the infinite summation (7). Using mathematica to find the root of (8), we establish the relationship between the chemical potential and temperature of the system.

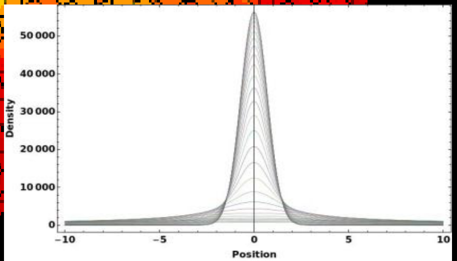



We now desire to establish the percent of molecules which are in their ground states as a function of temperature. First we must define dimensionless variables for the chemical potential and the temperature, both denoted with tilde above m and T respectively. We define the variable j as an index which relates to reduced temperature such that reduced temperature equals $j^2$. We wish to establish what percentage of molecules are in their ground state. Substituting dimensionless quantities into (3)we obtain (9), where i is the energy state and j is the index. We now plot the occupancy of particle in the zero energy state over the total number of particles (condensate fraction).



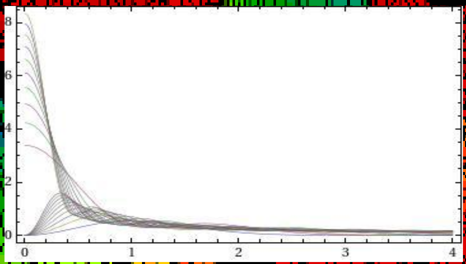

We notice that when the reduced temperature is at or near absolute zero, nearly all the particles are in their ground states. The instantaneous rate of change is which the condensate fraction undergoes is negative and constant until reaching a reduced temperature of approximately 10,000, which is the critical temperature when BEC begins occurring. Then the condensate fraction increases until converging to approximately zero.

We consider the density profile (10) which is the number of molecules occupying 1-D positions. We define dimensionless quantities for position, wave-function, and density profile, all denoted with tildes above their symbols. (10) now is converted to (11). Since the infinite summation cannot be directly evaluated we establish that the upper limit of 1000 is sufficient. We now plot the 1-D density profile for BEC.

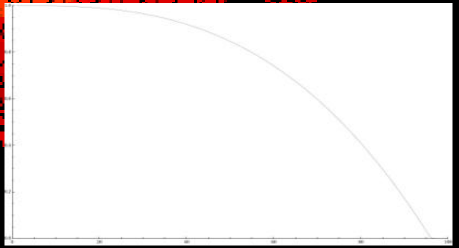



## Establishing the Density Profile of BEC in 3-D Harmonic Potential:

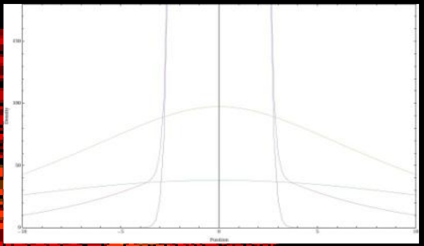
The potential trap used to cool gases to nearly absolute zero is nearly identical to simple harmonic oscillator potential given by (12). Applying (12) the Schrodinger equation we obtain a solution given by (13). Plotting these wave-functions we see the behavior of the solution for different energy quantum numbers. The highest peak of the plot is from the lowest energy quantum number n=0. The energy quantum numbers range from n=0 to n=20.



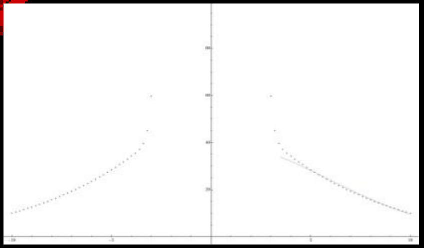

Again, we desire to establish the relationship between the chemical potential and temperature. The expected number of particles in an energy state n for BEC is given by (14). So we know the total number of particles in the system can be expressed by (15). We define dimensionless quantities for chemical potential and temperature and denote them by tildes above the original symbols mu and T, converting (15) to (16). For large values of n=n_l, we can simplify the final fraction in (16) using (17). Further simplification shows the last term in (17) can be expressed as (18). After thorough algebraic evaluation we finally establish the expression (19). (19) allows us to use mathematica to find the roots assuming we have 1,000,000 particles in the system. The condensate fraction can also be established. We plot the result.

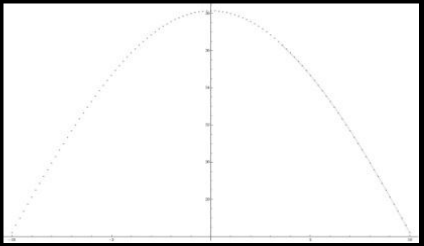


We now consider the density profile in 3-D which is given by (20). Recalling Unsold's Theorem (21), and defining dimensionless quantities for radius, and the density, denoted by tildes above the originial symbols, we establish the dimensionless 3-D density profile which is given by (22). The upper limit is originally infinite so we establish values of n which nearly all of the particles are accounted for by referring to the condensate fraction. We now graph the density profile (density vs. radius) for index values of j=1,50,99,148.



Notice the density does not include the entire range of values. Initially the results seem flawed. It seems our expression is losing particles, but upon closer inspection we realize that particles are not being lost, they are simply spred out over the lower values of the density. To further verify our conclusion, we decide to fit the tails of the graphs using mathematica. Understanding that the curves distribute Gaussian behavior, we use the general Gaussian equation to fit each tail. We plot these fits for index values of j=50, and j= 148, respectively.





Integrating these curves we find the areas under each curve to be 975,148 and 930,249, respectively. Nearly all the original 1,000,000 particles which make up the system are accounted for. These results further verify the validity of our density profile.