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Chaos and stability in the vicinity of a Jovian planet

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It has been widely known that the influence of large bodies (the Sun, the terrestrial and the giant planets) leads to the chaotic dynamics of the orbits of smaller members (asteroids, comets, and interplanetary dust) of the solar system. The orbits of these small bodies are known to undergo large changes on geological time scales. However, it is a matter of great speculation whether the orbits of the terrestrial planets are also chaotic over a long time scale due to the influence of the neighboring giant planets and the Sun.

Data Sheet

Sun

1. The Sun is about 4.57 billion years old, and it will last for about 5.5 billion years more.

2. Mass = 1.9891E30 kg

Jupiter

1. Mass = 1.8986E27 kg

2. Orbital speed = 13.07 km/s = 1.307E4 m/s

3. Semi-major axis = 778,547,200 km (around 5.20 AU)= 7.7855E11 m

4. Orbital period = 4331.572 days = 3.7425E8 s

Mars

1. Mass = 6.4185E23 kg

2. Orbital speed = 24.077km/s = 2.4077E4 m/s

3. Semi-major axis = 227,939,100 km (around 1.52 AU) = 2.2794E11 m.

4. Orbital period = 686.971 days = 5.9354E7 s.

Gravitational Constant

G = 6.673E-11 $m^3kg^{-1}s^{-2}$

Introduction

In the early 1600s, Johannes Kepler discovered and formulated the laws of planetary motion which was further simplified later by Sir Isaac Newton in his law of universal gravitation. In this law, Newton predicted that the force on each planet is simply the sum of the gravitational forces from the sun and all of the other planets in the solar system. In vector notation, this is expressed as:

$$m_i \frac{d^2\boldsymbol{r_i}}{dt^2} = Gm_i \sum_{j\neq i} m_j \frac{\boldsymbol{r_j} - \boldsymbol{r_i}}{\left|\boldsymbol{r_j} - \boldsymbol{r_i}\right|^3} \tag{1}$$

where G is the gravitational constant, m values are the masses, and $\boldsymbol{r}$ values are their position vectors in space (Holman). For a two-body system (the sun and a planet), there is a simple, elegant solution. However, the presence of a third body (or in the case of the solar system, the sun, the eight major planets, and many other minor bodies) allows no simple solution to these simple equations. Numerical analysis is known to be the best approach in this regard.

Chaos in the solar system

In everyday life, the word *chaos* is considered to be a synonym for disorder, commotion, complete randomness, etc. However, in science, chaos describes *the irregular behavior that can occur in deterministic dynamical systems,* i.e., systems described by ordinary differential equations free of random external influences. Chaotic systems are extremely sensitive to initial conditions, such as orbital radius, orbital speed, etc. Numerical analysis has shown that infinitesimal perturbations of the initial conditions for a deterministic dynamical system may lead to *exponential* variations of the orbits over a long period of time:

$$|\delta\boldsymbol{R}(t)| \approx |\delta\boldsymbol{R_0}|e^{\lambda t} \tag{2}$$

$$\ln|\delta\boldsymbol{R}(t)| \approx \ln|\delta\boldsymbol{R_0}| + \lambda t \tag{3}$$

λ is known as the Lyapunov exponent, and the maximum value for λ is obtained by projecting the time to infinity:

$$\lambda_{max} = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\delta \boldsymbol{R}(t)|}{|\delta \boldsymbol{R_0}|} \tag{4}$$

The time taken for the trajectories of a chaotic system to vary by a factor of e is called the *Lyapunov time*, defined by $1/\lambda$. Thus, the least possible time for a dynamical system to be chaotic is given by $1/\lambda_{max}$.

Chaos in the solar system is associated with gravitational resonances. The simplest case of a gravitational resonance occurs when the orbital periods of two planets are in the ratio of two small integers, e.g., 1:2, 3:5, etc. According to Kepler's Third Law of Planetary Motion, $T^2 \propto r^3$, any change in the time period leads to the change in the orbital radius. The long term dynamics of the planetary system is the dynamics of gravitational resonances. Within the last two decades, it has been understood that the orbits of many of the small members of the solar system (asteroids, comets, dust particles), subjected to the combined gravitational perturbations of the major planets, are chaotic and unstable on million-year time scales. Recent attempts have now been made to comprehend the chaotic dynamics involved in the orbits of the terrestrial planets due to the influence of the Jovian planets nearby (Holman).

Numerical approach to the theory of chaotic dynamics among major planets

In the last two decades remarkable advances in digital computer speed, the development of new numerical techniques, and the application of modern nonlinear dynamics techniques and chaos theory to classical problems of celestial mechanics have led to the discovery and exploration of a number of examples of dynamical chaos in our solar system. Scientists have investigated the orbital evolution of planetary orbits on giga-year time scales via several numerical simulations. These have led to an interesting conclusion that the orbits of the planets themselves evolve chaotically, the characteristic Lyapunov time being 5-10 million years. One such analytic theory of the chaos among the three Jovian planets - Jupiter, Saturn, and Uranus - confirms the numerical estimate of the escape time of Uranus to be 10^{18} years, substantially longer than the lifetime of our sun.

Although the numerical simulations indicate chaos in planetary orbits, in a qualitative sense the planetary orbits are stable as the planets remain near their present orbits over the lifetime of the sun. However, the presence of chaos implies that there is a finite limit to how accurately the positions of the planets can be predicted over long times. Takashi Ito discussed several properties that may be responsible for the long term stability of our solar system (Holman 12343). Among these, the difference in *dynamical separation* between terrestrial and Jovian planetary subsystems seems to be quite interesting and important. The terrestrial planets have smaller masses, shorter orbital periods, and wider dynamical separation. They are strongly perturbed by the Jovian planets, which have larger masses, longer orbital periods, and narrower dynamical separation. As a subsystem, the Jovian planets are not perturbed by any other massive bodies.

<u>The computer code</u>

Using FORTRAN 77, I developed a code with the help of Dr. Martin Veillette that would predict any chaotic behavior for a three body system, namely Mars, Jupiter, and the Sun, if projected to infinite time. A numerical analysis of Newton's universal law of gravitation (Equation 1) was used in the code. The final outcome was based on the three equations that described the acceleration, velocity, and position of the bodies all depending on time:

$$a_{ni}(p * \Delta t) = \sum G\, m_s * \frac{(r_{si} - r_{ni})(p * \Delta t)}{\left|\sum_j \left(r_{nj} - r_{sj}\right)^2\right|^{3/2}} \tag{5}$$

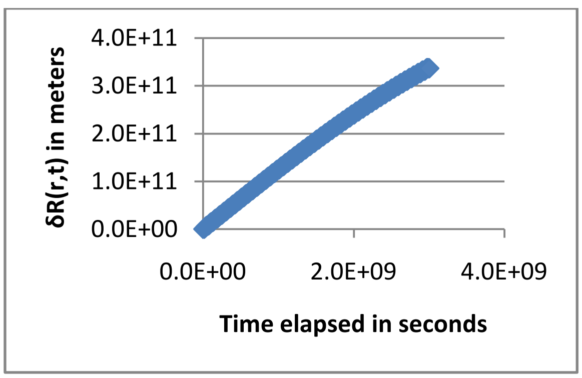
$$v_{ni}(\Delta t) = v_{ni}(0) + a_{ni}(0) * \Delta t \quad (6)$$

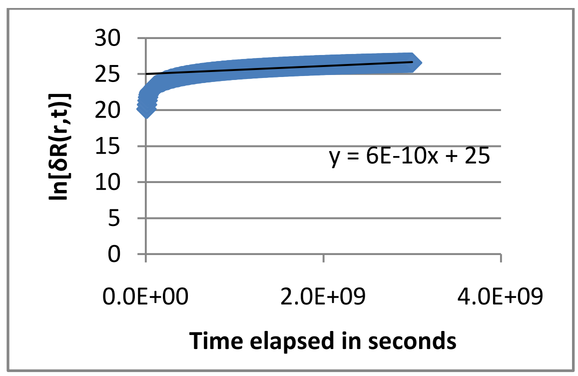
$$r_{ni}\big((p+1) * \Delta t\big) = r_{ni}(p * \Delta t) + v_{ni}(p * \Delta t) * \Delta t + \frac{1}{2} a_{ni}(p * \Delta t) * (\Delta t)^2 \quad (7)$$

A copy of the original code is attached at the end of this paper to understand the concept underlying the code. The system is initially taken to be two-dimensional (planar) with the known orbital radii and orbital speeds for the two planets just to check the validity of the code. Evaluation of the total energy, linear momentum, and angular momentum showed that the three values remain constant as time progresses indicating that the system is deterministic.

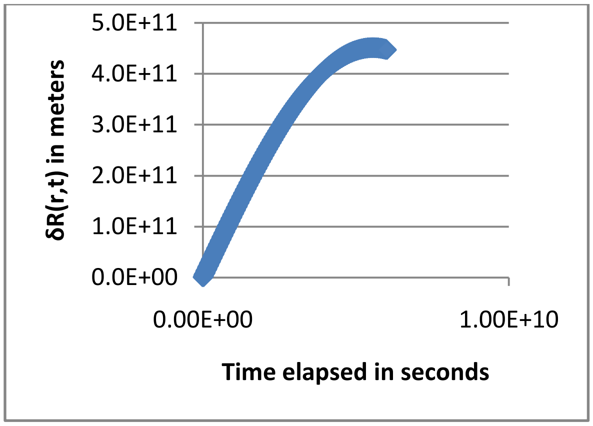
Graphical interpretation

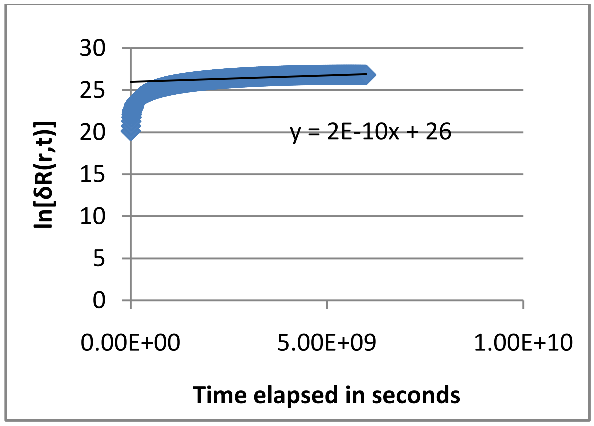
As mentioned earlier, it is expected to see chaotic behavior for our three-body system if it has time period ratios of comparably small integer values. We start with the ratio of 1:6 (which is the original time ratio of Mars and Jupiter in reality) to check whether the system shows any chaotic behavior over a long time span. The results are shown below:



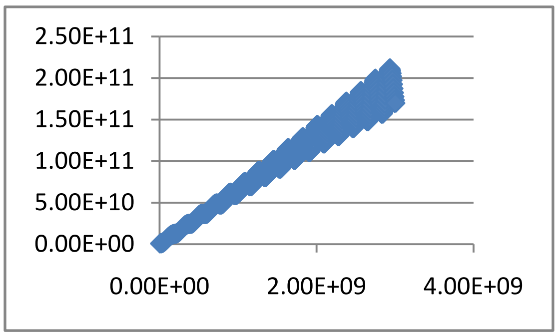


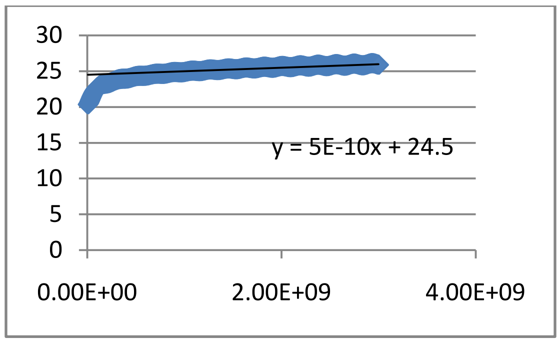
The above two figures are of time ratio of 1:6 predicted for 50 revolutions of Mars around the Sun. The divergence curve looks somewhat linear which is expected in the beginning for an exponential curve. However, the curve for $\delta R(t)$ seems to be reaching saturation point which indicates that the curve might not diverge for greater revolutions. Just to confirm, we projected it for 100 revolutions:



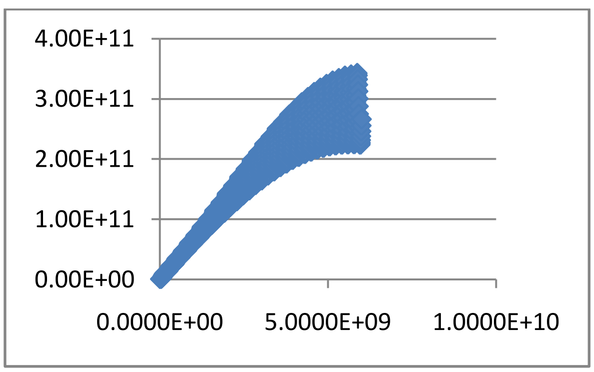


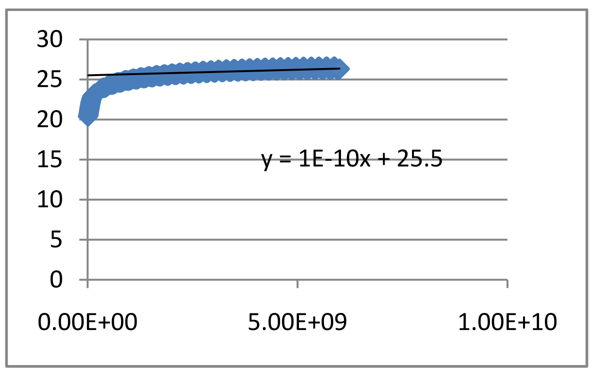
Next, we tried with the ratio of 1:2, and observed what the theory suggested for gravitational resonance:





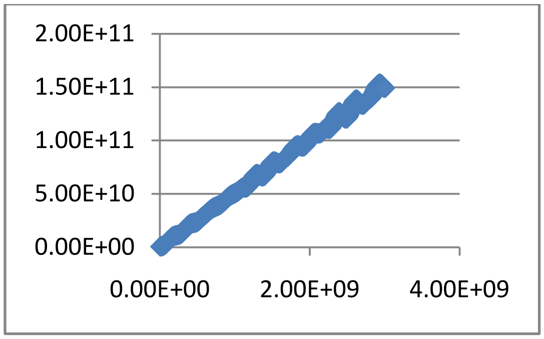
The above graphs for 50 revolutions seemed reasonable and promised something good for greater revolutions. So, we projected it for 100 revolutions:

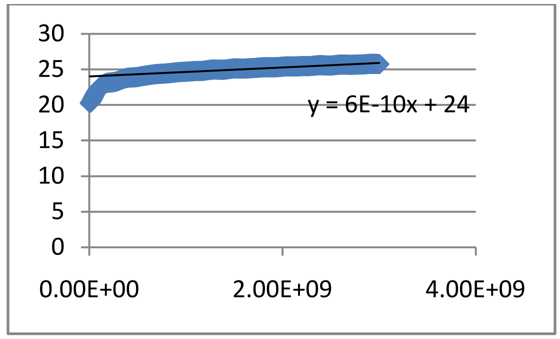




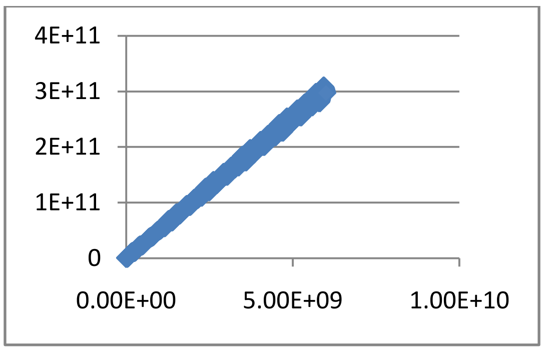
The result was not what we expected it to be. It seemed like this system would fail to be chaotic in the long run. However, it may be too early to make any predictions. So, we tried two more pairs of time ratios.

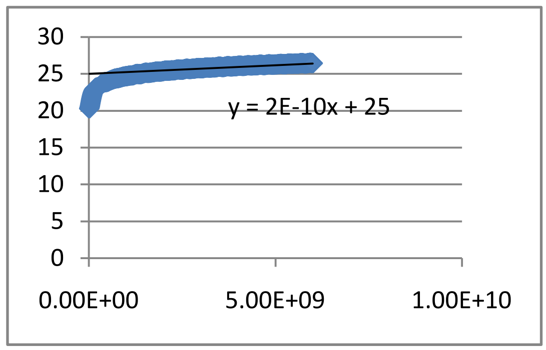
Time ratio = 3:5. Revolutions = 50





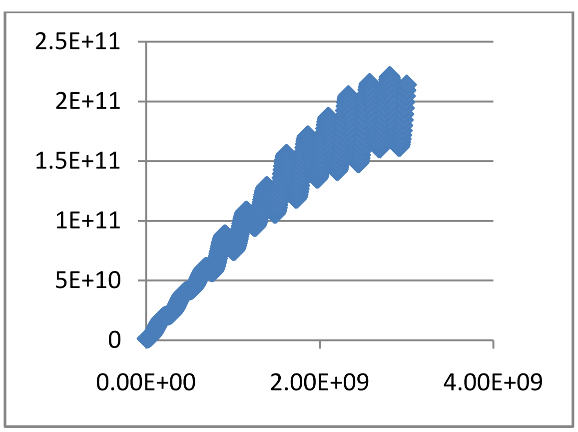
Time ratio = 3:5, Revolutions = 100

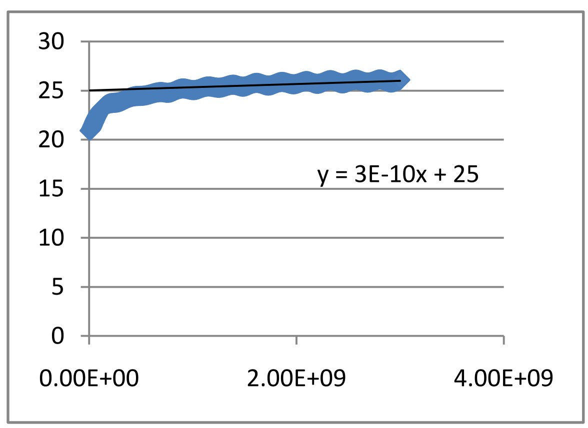




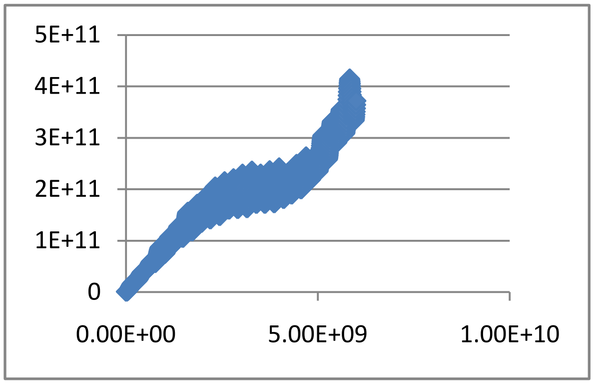
The above results for the time ratio 3:5 surely showed some improvement in the results as it seemed to be following the pattern of exponential increase in the separation between the trajectories.

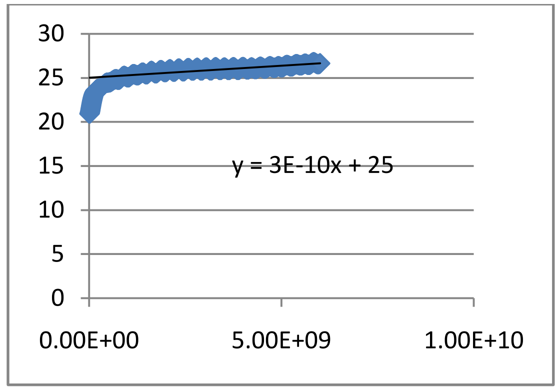
Time ratio = 2:3, Revolutions = 50





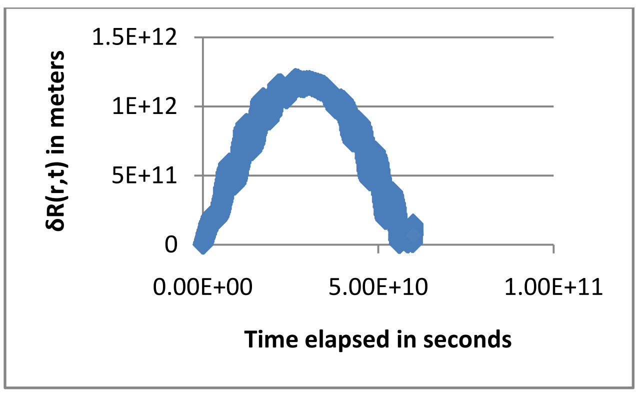
Time ratio = 2:3, Revolutions = 100

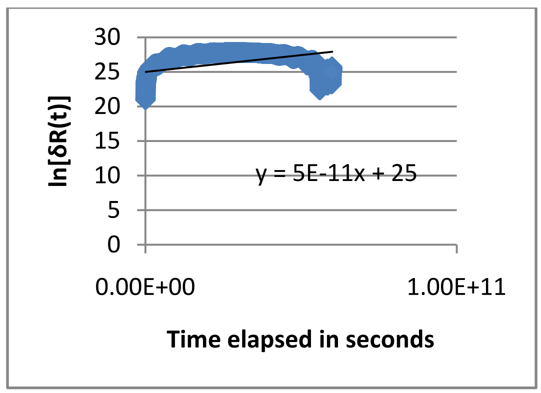




The above graph of time ratio 2:3 for 100 revolutions is very interesting in the sense that it showed some increasing divergence property at the very end. Perhaps, it will now be interesting to project one of the time ratios of the system for 1000 revolutions or more to see what happens.

Time ratio = 2:3, Revolutions = 1000





The result seemed to be quite disappointing at first. However, if we closely notice, we would see that the very end part of the *δR(t) vs t* graph promises a trend of increased divergence in trajectories once again. In fact, the periodic oscillations are predicted to show *increasing peak values* if the time is projected to infinity as *required* for a system to be proven chaotic according to Equation 4.

Future work

There were several impediments to our research as far as calculating gigantic amount of raw data was concerned. First, the 32 bit computer memory seemed to be inefficient in reducing error propagation for a large time-scale projection such as this. Also, it took around 8 hours to examine relatively *short* cases of 1000 revolutions, and it was difficult to keep the multi-purpose laptop machine busy for running code only while I had other coursework to deal with. However, the results obtained from our research promised good results if projected to at least a million year scale. To make things a little bit more interesting, the time step-size could be adjusted as well even though I feel that I used relatively short time step-size of 600 seconds (10 minutes) compared to the orbital period of Mars of 687 days.

Conclusion

This research helped me learn about the gravitational chaos theory in details. Moreover, I have considerably improved my skill in developing FORTRAN code, and analyzing a large amount of raw scientific data. Last, but not the least, this research taught me how to be patient while doing numerical analysis.

References

Arnett, Bill. "Mars". July 16, 2006. http://www.nineplanets.org/mars.html

Arnett, Bill. "Jupiter". July 16, 2006. http://www.nineplanets.org/jupiter.html

Arnett, Bill. "The Sun". July 16, 2006. http://www.nineplanets.org/sol.html

Briggs, Helen. *Planet-hunters set for big bounty*. Sunday, February 17, 2008. http://news.bbc.co.uk/2/hi/science/nature/7249884.stm

Holman, Matthew, Renu Malhotra, and Takashi Ito. *Chaos and Stability of the Solar System.* PNAS. Volume 98, No. 22. (October 23, 2001), pp. 12342-12343.