Chladni lines

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Contents

1 Introduction

Ernst Florens Friedrich Chladni was a German physicist who lived from 1756 to 1827.[4] He would do demonstrations in which he would sprinkle sand on a metal then by using a bow would let the plate resonate and the sand would collect at the nodes.[4] Nowadays this same experiment is easily reproduced but instead of using a bow to resonate the plate we use a signal generator and a electromagnetic shaker. The two plates that are most often used are a square and circular plates. Both these plates lend themselves to being solved easier because of their highly symmetrical shapes. For non-symmetrical systems one must use computer software in order to get the nodes and solve the partial differential equation known as the wave equation.

The experiment that was proposed looked at the nodal patterns observed on both a square and circular plate. Then to try and find an equation that would explain the motion of the plates.

There are three different types of theories that can be used to describe the motion of vibrating plates. The three theories are membrane, thin plate and thick plate. Each theory has different rules to explain the motion of the plate. Membrane Theory allows for the use of the wave equation to explain the motion of the system which simply means $\partial_{tt}u = c^2\nabla^2 u$.[5] Thin plate theory is more complicated and has less assumptions and it describes the motion of the plate as a fourth order differential equation $\nabla^4 w = \rho h \partial_{tt} w$.[5] Thick plate theory becomes very cumbersome and brings into account sheering because of the thickness of the plate. The plates used in this experiment never approach the thickness needed for this theory. The way to determine which theory should be used is by looking at the ratio between the thickness of the plate, h and the width of the plate, a.[5] The ratio used is $\frac{a}{h}$ if the ratio is bellow 8 then Thick plate theory holds, if the ratio is bellow above 8 and bellow 80 then thin plate theory holds, and if the ratio is above 80 then membrane theory is the proper theory to use.[5] After taking measurements of the plates the ratio is found to be within the same range for both the circular and square plate, 240-300. This range places the plates well within membrane theory. Which makes the main equation used to describe the motion of the system to be the wave equation.[5] Some other assumptions that go along with Membrane theory according to Ruggiero is

1. The boundaries are free from transverse shear forces and moments. Loads applied to boundaries must lie in planes tangent to the middle surface.

2. The normal displacements and rotations at the edges are unconstrained: that is, these edges can displace freely in the direction of the normal to the middle surface.

3. A membrane must have a smoothly varying, continuous surface.

4. The components of the surface and edge loads must also be smooth and continuous functions of the coordinates.

He goes on to explain that these four assumptions lead to two very interesting characteristics of membranes.

1. Membranes do not have any flexural rigidity, and therefore cannot resist any bending loads.

2. Membranes can only sustain tensile loads. Their inability to sustain compressive loads leads to the phenomenon known as wrinkling.

Those are the main assumptions of membrane theory.[5] The next section will be delving into the derivation and some solutions to the wave equation

2 Wave Equation

2.1 One dimensional wave equation

The wave equation has an easy derivation by using Hooke's Law and Newton's second Law. First you can just look at the force of a particle thats mass does not change with time using Newton,

$$F = \frac{\partial p}{\partial t} = m\frac{\partial^2 u(x,t)}{\partial^2 t} \tag{1}$$

Now using Hooke's Law we can write the Force on a particle connected to two other identical particles by springs where each particle is then connected to a new particle and so on. Each particle will be a distance h apart and the stiffness of the springs will be k. The Force on the particle located at $x+h$ connected by a spring to particles located at x and $x+2h$ can be written as

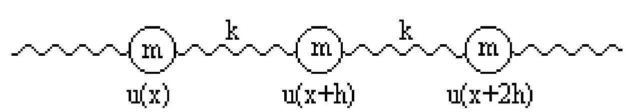

Figure 1: Particle located at $x+h$

$$\sum F = F_{x+2h} + F_x = k\{u(x+2h,t) - u(x+h,t)\} + k\{u(x,t) - u(x+h,t)\} \tag{2}$$

Now if we combine them still looking at the same particle

$$m\frac{\partial^2 u(x+h,t)}{\partial^2 t} = k\{u(x+2h,t) - 2u(x+h,t) + u(x,t)\} \tag{3}$$

By making a few more variables we can simplify the equation a little more. First variable is the number of particles N, the total length of the system is $L = Nh$, the total spring constant $K = \frac{k}{N}$, and total mass of the system is $M = Nm$.

$$\frac{\partial^2 u(x+h,t)}{\partial^2 t} = \frac{KL^2}{M}\frac{u(x+2h,t) - 2u(x+h,t) + u(x,t)}{h^2} \tag{4}$$

If we take the limit of $N \to \infty$ and $h \to 0$ now by just using the difference quotient definition of a derivative one gets

$$\frac{\partial^2 u(x+h,t)}{\partial^2 t} = \frac{KL^2}{M}\frac{\partial^2 u(x+h,t)}{\partial x^2} \tag{5}$$

2.2 Solution to the wave equation on a string

One method of answering a partial differential equation is to us separation of variables, that is to assume that the position of the particle $u(x,t)$ can be written as $X(x)T(t)$. From now on $\frac{\partial}{\partial x}$ will be written as ∂_x. The equation can now be written as the following.

$$\partial_{xx}X(x)T(t) - \frac{1}{c^2}\partial_{tt}X(x)T(t) = 0 \tag{6}$$

Since the function $T(t)$ does not depend on x it is considered a constant with respect to it, same for the function $X(x)$ with respect to t. Each can be pulled out of the other partial. Also the Trivial solution where either function is equal to zero is not the case we are looking at.

$$T(t)\partial_{xx}X(x) - \frac{X(x)}{c^2}\partial_{tt}T(t) = 0 \tag{7}$$

$$\frac{1}{X(x)}\partial_{xx}X(x) - \frac{1}{c^2 T(t)}\partial_{tt}T(t) = 0 \tag{8}$$

Now when looking at the previous equation it can be written as just simply two functions.

$$f(x) - g(t) = 0 \tag{9}$$

This has to be true for all values of x and t. The only way for that to be true is that each be equal to a constant. Let us call this constant λ_1.

$$\frac{1}{X(x)}\partial_{xx}X(x) = \lambda_1 = \frac{1}{c^2T(t)}\partial_{tt}T(t) \tag{10}$$

Now just looking at the function $X(x)$.

$$\partial_{xx}X(x) = \lambda_1 X(x) \tag{11}$$

The answer to this partial differential equation can be written simply.

$$X(x) = Ae^{(\sqrt{\lambda_1}x)} + Be^{-(\sqrt{\lambda_1}x)} \tag{12}$$

Now once again looking at the non-trivial solution so $A \neq 0 \neq B$. Looking first at λ_1 is positive and the boundary conditions of a string with fixed ends located at $\pm\frac{L}{2}$ So that means $X(\pm\frac{L}{2}) = 0$.

$$0 = Ae^{(\sqrt{\lambda_1}\frac{L}{2})} + Be^{-(\sqrt{\lambda_1}\frac{L}{2})} \tag{13}$$

$$0 = Ae^{-(\sqrt{\lambda_1}\frac{L}{2})} + Be^{(\sqrt{\lambda_1}\frac{L}{2})} \tag{14}$$

Now using one to substitute into the other we get

$$\begin{aligned}
A &= \frac{-Be^{-(\sqrt{\lambda_1}\frac{L}{2})}}{e^{(\sqrt{\lambda_1}\frac{L}{2})}} \\
A &= -Be^{-(\sqrt{\lambda_1}L)} \\
0 &= -Be^{-(\sqrt{\lambda_1}\frac{3L}{2})} + Be^{(\sqrt{\lambda_1}\frac{L}{2})} \\
-\sqrt{\lambda_1}\frac{3L}{2} &= \sqrt{\lambda_1}\frac{L}{2} \\
-\frac{3}{2} &= \frac{1}{2}
\end{aligned} \tag{15}$$

This shows that it is impossible λ_1 can not be positive. So now let us assume that $\lambda_1 = -k^2$ where k is a real number. We can write the answer to the partial differential equation a little different.

$$X(x) = A\cos{(kx)} + B\sin{(kx)} \tag{16}$$

Once again we use the same boundary conditions, and simplify.

$$
\begin{aligned}
0 &= A\cos{(k\frac{L}{2})} + B\sin{(k\frac{L}{2})} \\
0 &= A\cos{(-k\frac{L}{2})} + B\sin{(-k\frac{L}{2})} \\
0 &= A\cos{(k\frac{L}{2})} - B\sin{(k\frac{L}{2})}
\end{aligned} \tag{17}
$$

The only way for this to be true is if $B = 0$. Then we simply have the condition to not be trivial that $A \neq 0$. The only way this is possible is if $\cos{(k\frac{L}{2})} = 0$ This is only true if

$$k = \frac{n\pi}{L} \tag{18}$$

where n is odd. Now looking at $T(t)$ keeping in mind that $\lambda_1 = \frac{1}{c^2 T(t)}\partial_{tt}T(t)$

$$\partial_{tt}T(t) = c^2\lambda_1 T(t) \tag{19}$$

Once again this answer can simply be show to be

$$T(t) = A\cos{(\sqrt{\lambda_1}ct)} + B\sin{(\sqrt{\lambda_1}ct)} \tag{20}$$

The time period of an oscillation of this function can be written as $T = \frac{2\pi}{\sqrt{\lambda_1}c}$ this can be changed into the frequency by using the relationship that the time period is the reciprocal of the frequency $f = \frac{\sqrt{\lambda_1}c}{2\pi}$. The relation can be simplified by looking at the angular frequency.

$$\omega^2 = \lambda_1 c^2 \tag{21}$$

we can call $\lambda_1 = \frac{\omega^2}{c^2}$. This can be said through the same reasons previous just this time since there is another constant term c^2 that is why it was divided by it. The only conditions for time though was that $T(0) = 0$ this means that the cosine term gets removed and all that is left is the sine term. So we can now look at the position function.

$$u(x,t) = A\cos{\left(\frac{n\pi x}{L}\right)}\sin{(\omega t)} \tag{22}$$

That is the solution for a standing wave on a string with fixed ends, but the symmetry of a plate lends it self to being a string with unfixed ends.

All this changes is the boundary conditions for position. The new boundary conditions for $X(x)$ is $\partial_x X(x)$ at $x = \pm\frac{L}{2}$ is zero.

$$\begin{aligned}
0 &= -Ak\sin{(k\frac{L}{2})} + Bk\cos{(k\frac{L}{2})} \\
0 &= -Ak\sin{(-k\frac{L}{2})} + Bk\cos{(-k\frac{L}{2})} \\
0 &= Bk\cos{(k\frac{L}{2})} \qquad (23)
\end{aligned}$$

So the final results now looks like

$$u(x,t) = A\sin\left(\frac{m\pi x}{L}\right)\sin{(\omega t)} \qquad (24)$$

2.3 Wave equation in two dimensions

The wave equation can be looked at in two dimensions also instead of just one dimension as above. All that is needed is to add two more connections to the original particle. If the original two springs were connected along the x-axis then the two new springs will connect along the y-axis such that they will be orthogonal and have no interference with the springs along the x-axis. If we second pair of springs anywhere but the y or z-axis then they would have components that would add to the x-axis springs. let us say that the spring constant is the same in the x and y directions. The distance separating the particles is h_x and h_y each referring to there own axis. Using Hooke's Law once again we look at the particle located at $(x + h_x, y + h_y)$

$$\sum F = F_{x+2h_x} + F_x + F_{y+2h_y} + F_y \qquad (25)$$

This equation can be simplified the exact same way as the previous just substituting in the variables L for a new ones named L_x and L_y for the total length in their respective directions, and K for the spring constant in each direction.

$$\partial_{tt}u(x,y,t) = \frac{KL_x^2}{M}\partial_{xx}u(x,y,t) + \frac{KL_y^2}{M}\partial_{yy}u(x,y,t) \qquad (26)$$

The constant term in front is simply the speed at which the wave propagates through the medium squared. The partial derivatives can be written as simply the laplacian. So for second and higher dimensions the wave equation looks like the following.

$$\frac{\partial^2 u}{\partial t^2} = v_m^2\nabla^2 u \qquad (27)$$

2.4 Solution to the wave equation on a square plate

To answer this partial differential equation once again separation of variables will be the method employed. So assume that $u(x, y, t)$ can be written as $X(x)Y(y)T(t)$ so then it simplifies to three different differential equations. After a small bit of simplification we now have

$$\frac{1}{X(x)}\partial_{xx}X(x) + \frac{1}{Y(y)}\partial_{yy}Y(y) - \frac{1}{c^2T(t)}\partial_{tt}T(t) = 0 \tag{28}$$

All arguments will be identical to that in the one dimensional case. The only difference is that the three different parts of this differential equation do not equal all the same constant but instead each equals a constant that fallows this relation.

$$\lambda_x + \lambda_y + \lambda_t = 0 \tag{29}$$

From previous examples we can take a guess that λ_x and λ_y are negative and λ_t is positive. The same solution for the x direction and the y direction's solution is the same as the x. The only difference for the solution for the time function is what ω is equal to. In this case $\omega^2 = c^2(k_x^2 + k_y^2)$. Thus the answer for the two dimensional with non fixed edges is

$$u(x, y, t) = A\sin\left(\frac{n\pi x}{L_x}\right)\sin\left(\frac{m\pi y}{L_y}\right)\sin(\omega t) \tag{30}$$

That solution is if the origin is considered to be in the middle of the plate, but also this has restrictions on what n and m but if the origin is placed at the bottom left corner of the plate a different solution is seen with the restriction of only that n and m must be integers.

$$u(x, y, t) = A\cos\left(\frac{n\pi x}{L_x}\right)\cos\left(\frac{m\pi y}{L_y}\right)\sin(\omega t) \tag{31}$$

That solution works for any rectangular plate. The plate used in this experiment will be a square plate which leads to an interesting phenomena. Since L_x and L_y are indistinguishable, this implies that for a given n and m both of the following are correct answers.

$$u(x, y, t) = A\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{L}\right)\sin(\omega t) \tag{32}$$

$$u(x, y, t) = A\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right)\sin(\omega t) \tag{33}$$

So the answer to a square plate truthfully is a superposition of both. This leads to the position answer to being the following

$$u(x, y, t, \alpha) = \cos\alpha \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{L}\right) \sin(\omega t) \tag{34}$$

$$+ \sin\alpha \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi y}{L}\right) \sin(\omega t) \tag{35}$$

2.5 Wave equation in Polar coordinates

The wave equation is the same in all coordinates it is just the laplacian that is different. The laplacian in polar is

$$\nabla^2 f(\rho, \theta) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} \tag{36}$$

so once again separation of variable is the method that will be employed so the position function can be written as

$$u(\rho, \theta, t) = P(\rho)\Theta(\theta)T(t) \tag{37}$$

The reason to use polar coordinates is if you have a symmetry of a problem lends it self to being circular. In the last two sections what was looked at is a square plate. The next section is looking at the answer to the wave equation in a circular plate.

2.6 Solution to the wave equation on a circular plate

The wave equation simplifies to

$$0 = \frac{1}{P(\rho)\rho}\partial_\rho\left(\rho\partial_\rho P(\rho)\right) + \frac{1}{\Theta(\theta)\rho^2}\partial_{\theta\theta}\Theta(\theta) - \frac{1}{Tc^2}\partial_t T(t) \tag{38}$$

This equation can be rewritten as $0 = f(\rho, \theta) + g(t)$. Once again the only way for two functions that are independent of each other to add up to zero for all values is if they equal a constant. So $-f(\rho, \theta) = \frac{\omega^2}{c^2} = g(t)$ using the same reasons for the answer to the time dependent portion of function we have

$$T(t) = A\sin(\omega t) \tag{39}$$

Now just looking at the other portion of the function

$$\frac{-\omega^2}{c^2} = \frac{1}{P(\rho)}\partial_{\rho\rho}P(\rho) + \frac{1}{P(\rho)\rho}\partial_\rho P(\rho) + \frac{1}{\Theta(\theta)\rho^2}\partial_{\theta\theta}\Theta(\theta) \tag{40}$$

This equation can simplify to

$$0 = \frac{\rho^2}{P(\rho)}\partial_{\rho\rho}P(\rho) + \frac{\rho}{P(\rho)}\partial_\rho P(\rho) + \frac{\rho^2\omega^2}{c^2} + \frac{1}{\Theta(\theta)}\partial_{\theta\theta}\Theta(\theta) \tag{41}$$

For the second time while solving the same equation the logic of writing the equation as $0 = f(\rho) + g(\theta)$. The only way for this to hold true for all values of ρ and θ is if both are equal to 0. Say $f(\rho) = \alpha^2$ and that $g(\theta) = -\alpha^2$. The easier of the two to solve is the polar angle $\Theta(\theta)$

$$\partial_{\theta\theta}\Theta(\theta) = -\alpha^2\Theta(\theta) \tag{42}$$

$$\Theta(\theta) = A\cos(\alpha\theta) + B\sin(\alpha\theta) \tag{43}$$

The answer now can be simplified by adding the boundary condition that every 2π the function $\Theta(\theta)$ has to be the same. This will ensure that at the position on the plate there is only one answer. This implies

$$A\cos(\alpha\theta) + B\sin(\alpha\theta) = A\cos(\alpha\theta + 2\pi\alpha) + B\sin(\alpha\theta + 2\pi\alpha) \tag{44}$$

$$\begin{aligned} A\cos(\alpha\theta) + B\sin(\alpha\theta) =& A\left[\cos(\alpha\theta)\cos(2\pi\alpha) - \sin(\alpha\theta)\sin(2\pi\alpha)\right] \\ &+ B\left[\sin(\alpha\theta)\cos(2\pi\alpha) + \sin(2\pi\alpha)\cos(\alpha\theta)\right] \end{aligned} \tag{45}$$

Other than the trivial solution this can be broken up into two different equations one having parts with the same constant equal to each other.

$$\cos(\alpha\theta) = \cos(\alpha\theta)\cos(2\pi\alpha) - \sin(\alpha\theta)\sin(2\pi\alpha) \tag{46}$$

$$\sin(\alpha\theta) = \sin(\alpha\theta)\cos(2\pi\alpha) + \sin(2\pi\alpha)\cos(\alpha\theta) \tag{47}$$

These can only be true if all of the following is true since sine and cosine are linearly independent.

$$\cos(\alpha\theta) = \cos(\alpha\theta)\cos(2\pi\alpha) \tag{48}$$

$$\sin(\alpha\theta) = \sin(\alpha\theta)\cos(2\pi\alpha) \tag{49}$$

$$0 = \sin(\alpha\theta)\sin(2\pi\alpha) \tag{50}$$

$$0 = \sin(2\pi\alpha)\cos(\alpha\theta) \tag{51}$$

once again this must be true for all θ so the four equations simplify to two that both must be true at the same time.

$$1 = \cos(2\pi\alpha) \tag{52}$$
$$0 = \sin(2\pi\alpha) \tag{53}$$

These both are true when α is an integer. Now if we say n instead of α to use a more fitting variable. Now the answer can be written as

$$\Theta(\theta) = A\cos(n\theta) + B\sin(n\theta) \tag{54}$$

All the linear combination does in this case is show the general solution but the difference between all solutions is just rotating the plate so one solution out of the infinite can be taken. So now the solution is simply

$$\Theta(\theta) = A\cos(n\theta) \tag{55}$$

Now all that is left to solve is the radial part of the problem. Which still using the fact that α has to be an integer we keep using n instead.

$$\frac{\rho^2}{P(\rho)}\partial_{\rho\rho}P(\rho) + \frac{\rho}{P(\rho)}\partial_\rho P(\rho) + \frac{\rho^2\omega^2}{c^2} = n^2 \tag{56}$$

With just a little rearranging this becomes

$$0 = \rho^2\partial_{\rho\rho}P(\rho) + \rho\partial_\rho P(\rho) + \frac{\omega^2}{c^2}\rho^2 P(\rho) - n^2 P(\rho) \tag{57}$$

by making a substitution of variables this equation can be transformed into a form with a more obvious answer. The substitution is let $\rho = \frac{c}{\omega}x$ and just say $P(\rho) = Y(x)$ this change of variable leads to several other changes too

$$\frac{d}{d\rho} = \frac{d}{d\left(\frac{c}{\omega}x\right)} = \frac{\omega}{c}\frac{d}{dx} \tag{58}$$
$$\frac{d^2}{d\rho^2} = \frac{d}{d\left(\frac{c^2}{\omega^2}x^2\right)} = \frac{\omega^2}{c^2}\frac{d^2}{dx^2} \tag{59}$$

Now to plug in the substitution

$$0 = \frac{c^2}{\omega^2}x^2\frac{\omega^2}{c^2}\partial_{xx}Y(x) + \left(\frac{c}{\omega}x\right)\left(\frac{\omega}{c}\partial_x Y(x)\right) + x^2Y(x) - n^2Y(x)$$
$$0 = x^2\partial_{xx}Y(x) + x\partial_x Y(x) + \left(x^2 - n^2\right)Y(x) \tag{60}$$

The answer to this partial differential equation is a linear combination of two Bessel functions. The two Bessel functions that have been chosen to be the answer is the $J_n(x)$ and the $Y_n(x)$. They each are defined as the following

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s} \tag{61}$$

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} \tag{62}$$

The answer to the differential equation once the variables are re substituted is

$$P(\rho) = AJ_n\left(\frac{\omega\rho}{c}\right) + BY_n\left(\frac{\omega\rho}{c}\right) \tag{63}$$

This is supposed to represent the physical system of a circular membrane so there are two boundary conditions to consider. The first is that the circular membrane is continuous at $\rho = 0$. This simple condition simplifies the radial piece because $Y_n(0)$ is not defined. So the only way to be continuous is if $B = 0$. The second boundary condition is that $\partial_\rho P(\rho)|_{\rho=R} = 0$. Let $k = \frac{\omega}{c}$ This leads to the following

$$\partial_\rho P(\rho)|_{\rho=R} = A\partial_\rho \left. J_n(k\rho)\right|_{\rho=R} = 0 \tag{64}$$

$$0 = A\frac{1}{2}k(J_{-1+n}(kR) - J_{1+n}(kR)) \tag{65}$$

$$J_{-1+n}(kR) = J_{1+n}(kR) \tag{66}$$

This is only true for certain values of k these values depend on n and since the Bessel function is periodic there is not just one k for each n there is an infinite number of answers so we will call a second variable m such that it is the mth time there is an answer to the solution. From now on k will be referred to as $k_{(n,m)}$. This leaves the Radial answer to be

$$P(\rho) = AJ_n(k_{(n,m)}\rho) \tag{67}$$

This leaves the solution to the wave equation as

$$u(\rho, \theta, t) = AJ_n(k_{(n,m)}\rho)\cos(n\theta)\cos(ck_{(n,m)}t) \tag{68}$$

3 Experimental Measurements

The set up used was a wave generator hooked up to a speaker that vibrated a aluminum plate that was sprinkled with sand. A camera was placed above the plate to take pictures at different frequencies. Both plates were started at low frequencies and scanned through to higher frequencies until a resonance pattern is shown.

Figure 2: Lab setup

After the nodal patterns were observed and documented. Then the pictures were transfered over to a computer where Mathematica was used to compare the different nodal patterns to the theoretical results. Through comparing the patterns to the ones produced by the computer a proper n and m can be determined.

3.1 The Square plate

The first patterns were observed at the corresponding frequencies. All frequencies bellow are measured in Hz. Figures 3a -10c are pictures of different nodal patterns. While figures 11 -13 are a compilation of the answers found on Mathematica and the picture of the plate taken. The numbers under those pictures state the $(n, m)\alpha$ in there respective order. In order to find exactly which picture matched which nodal number Mathematica was used and since there were several different patterns for each n and m it was customary to guess and check since there was no definable way to find the α, the summation constant.

3.2 Circular Plates

The patterns seen with the circular plate were easier to identify which of the wave nodes it should be. Looking at the Mathematica program there were only one distinct pattern for every n and m. Each m another radial node is created and each additional n creates another angular node. Unfortunately the angular nodel patterns that were observed in the program were not the same observed during the experiment. Figures 14a- 18a are pictures of the circular plate and the frequencies that created them. The figures 19 and 20 show overlapping images of the caclulated nodes compaired to the nodes observed. All those have $n = 0$ and under is the value of m.

4 Data Analysis

To analyze the data using the fact that there is a relation between the frequency and wave number. This relation is different for each plate. As a recap the relationship for a square plate is $\omega^2 = c^2(k_x^2 + k_y^2)$ and the relationship for the circular plate is $k_{(n,m)} = \frac{\omega}{c}$. with both of those relations the easiest way to analyze the data is to make a graph comparing $\frac{f}{f_o}$ to a $\frac{k}{k_o}$,where f_o stands for some constant frequency and k_o is the wave number that corresponds to f_o, this relation in both cases can get rid of constants. For the square plate the relation expected is

$$\frac{f}{f_o} = \sqrt{\frac{n^2 + m^2}{n_o^2 + m_o^2}} \tag{69}$$

and the circular plate has a much simpler relation.

$$\frac{f}{f_o} = \frac{k_{(n,m)}}{k_{(n_o,m_o)}} \tag{70}$$

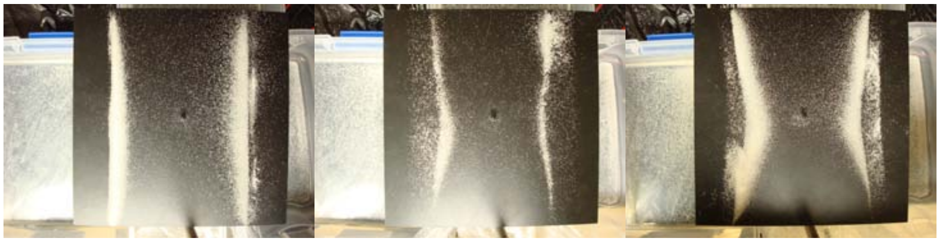

Figure 3: 100.0, 104.6, 110.4

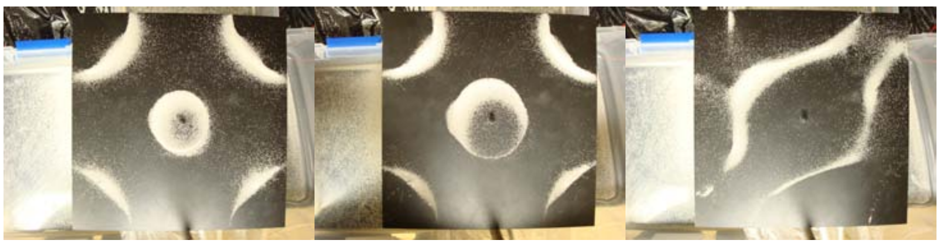

Figure 4: 174.3, 187.3, 243.3

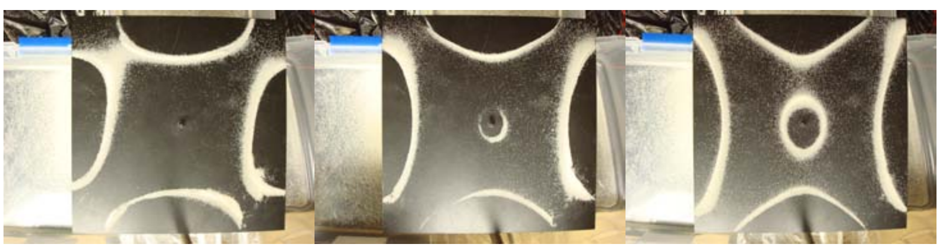

Figure 5: 279.0, 313.5, 330.1

Figure 6: 380.7, 399.6, 407.5

The square plate graph is shown in figure 21. This graph shows $\log\left(\frac{f}{f_o}\right)$ vs $\log\left(\sqrt{\frac{n^2+m^2}{n_o^2+m_o^2}}\right)$ along with the relation above the graph should have a slope

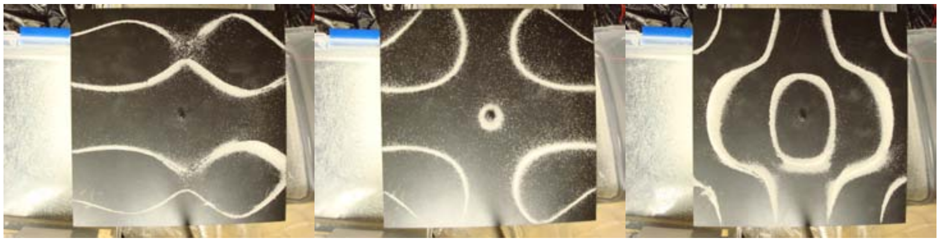

Figure 7: 443.3, 467.0, 525.3

Figure 8: 535.5, 538.4, 559.2

Figure 9: 725.0, 847.3, 924.1

Figure 10: 940.7, 1028.8, 1656.5

of 1. The slope of the graph is found to be 2.

With the circular plate the graph is figure 22. This graph shows $\frac{f}{f_o}$ vs

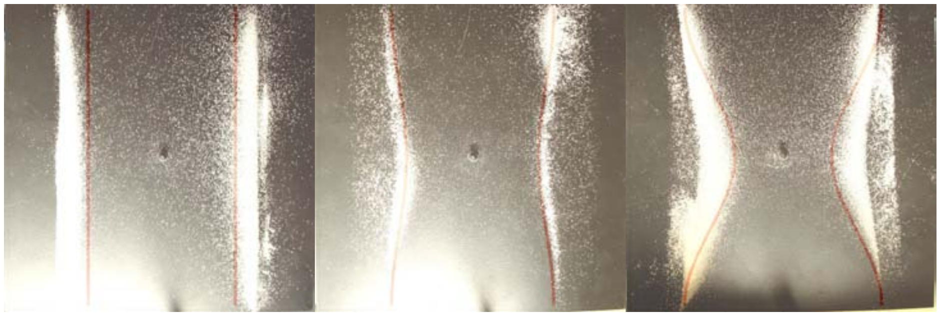

Figure 11: $(2,0)0, (2,0)\frac{19\pi}{20}, (2,0)\frac{17\pi}{20}$

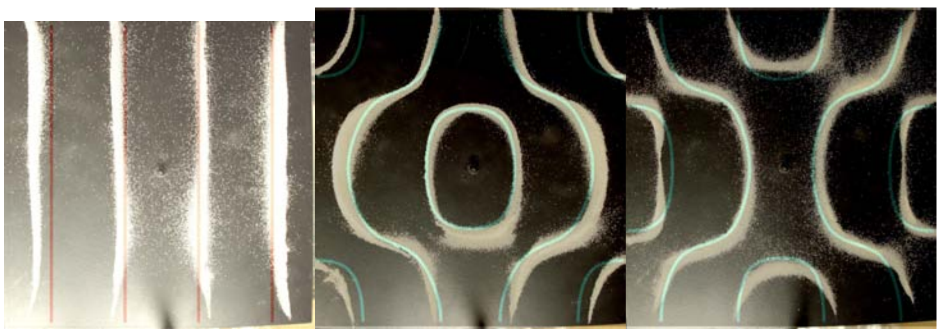

Figure 12: $(4,0)0, (2,4)\frac{8\pi}{20}, (2,4)\frac{12\pi}{20}$

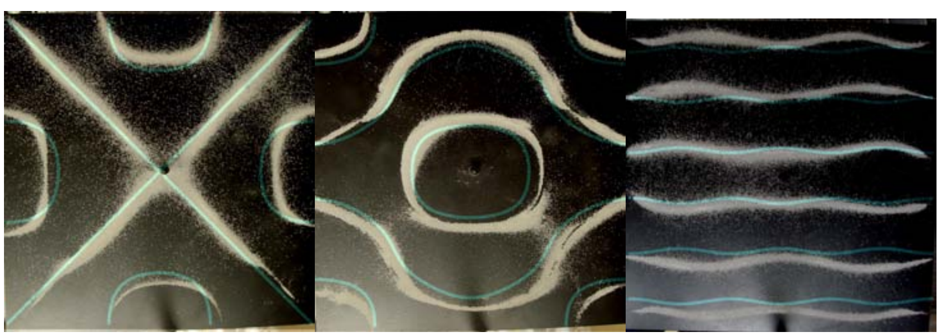

Figure 13: $(2,4)\frac{15\pi}{20}, (2,4)\frac{3\pi}{20}, (6,0)\frac{1\pi}{20}$

$\frac{k_{(0,m)}}{k(0,1)}$ once again the slope this time should be 1. The error is huge this time with a slope of 8.2.

Figure 14: 93.59, 135.38, 288.13

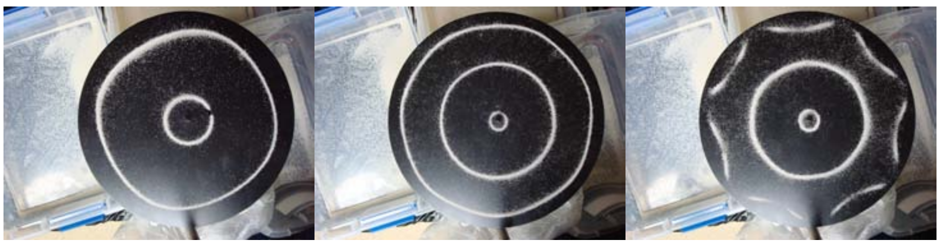

Figure 15: 317.79, 795, 806

Figure 16: 1520, 1569, 2510

Figure 17: 3816, 4973, 5360

5 Conclusion

The Mathematica simulation and solutions used for the answers were the same used by several different sources. The solution for the circular plate

Figure 18: 7041

Figure 19: 1, 2, 3

Figure 20: 4, 5

was been extremely different than what the experiment showed. This may have been because of several deformations of the plate the pictures took show a 1% eccentricity, this may simply be a trick of taking the picture because if either the camera or the plate was not level then a small eccentricity would be shown. From physical observation of the plate small flat sides were found meaning the plate was not a perfect circle. Something that may be the

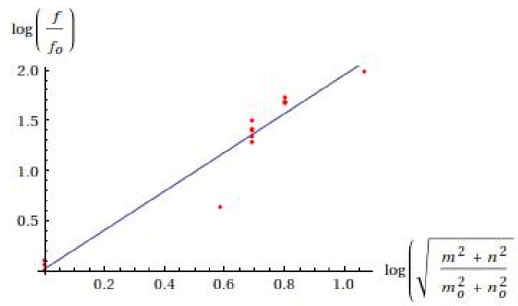

Figure 21: $y = 0.02 + 1.96x$

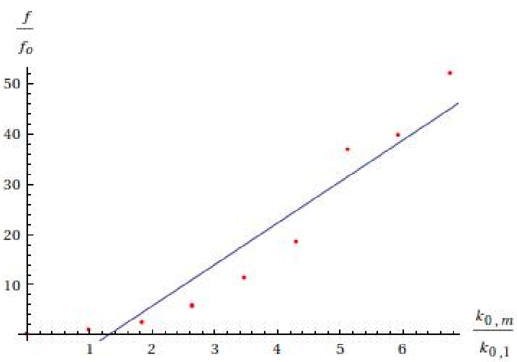

Figure 22: $y = -10.7 + 8.2$

most important is the fact that there is a hole cut slightly off center for the vibrator pin to be placed for examining the nodal patterns of a circle vibrated off center. This hole may of interfered with the addition of the waves that were propagated across the plate. All these reasons seem moot for the error found between the Mathematica model and the observed patterns was humongous. No explanation found seemed to justify why the angular nodes were not the same at all. The only explanation that even began to explain it was that the symmetry of the system was not broken thus the patterns did not emerge.[2] The reason why the frequencies did not match up between both plates and the expected node is probably because of the use of membrane theory. Looking back to the beginning to the characteristics of a membrane it says that membranes have no flexural rigidity, and can not sustain compressive loads. Both of these characteristics are lacking in the aluminum plate. The plate does not compress easily nor does it bend easily. If the plate bended easily the experimental setup would have had to change drastically. So the theory that probably would have held a more reasonable answer would have been thin plate theory. But the solutions to the wave pattern would be very similar it would simply be the frequency at which they are found that would change. Further investigation into thin plate theory is needed to make any more conjectures.

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