Formulating a Theory for the GSI Anomaly

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1 Introduction

Radioactive decay has been a widely-studied phenomenon, it is a process through which unstable nuclei get rid of the excessive energy by emitting ionizing particles and radiation. From basic physics, we are aware of alpha, beta and gamma decays (See Wikipedia: Alpha, Beta and Gamma Decays):

1. Alpha Decay
$$^{238}_{92}U \rightarrow ^{234}_{90} Th +^{4}_{2} He$$
2. Beta Decay (2 types)
$$^{137}_{55}Cs \rightarrow ^{137}_{56} Ba + e^{-} + \upsilon_e$$
$$^{22}_{11}Na \rightarrow ^{22}_{10} Ne + e^{+} + \upsilon_e$$
3. Gamma Decay - No change in nuclear composition of the mother nucleus, but spontaneous emission of gamma radiation.

It would be redundant to explain the various properties of these decays in this paper; one thing that all of these decays have in common is that the number of nuclei decay exponentially as a function of time. However, recent study of the decay of hydrogen-like $^{140}Pr^{59+}$ and $^{142}Pm^{60+}$ baffled scientists with the existence of an oscillating function riding on top of the exponential decay (See Physics Letters B 664 (2008) [4] Pages 162-168). Several theories have been proposed to explain the phenomenon with possibilities ranging from neutrino mixing to the formation of quantum beats due to similar energy eigenstates of the mother nuclei. None of the explanations have proven to be conclusive so far even though the scientific community tend to be more favorable towards the latter. My aim in this paper is to use basic quantum mechanics to understand the cause of the oscillations and see which one of the proposed theories seem to be the 'best fit' in understanding the GSI anomaly.

1.1 Some Experimental Background

The experiment was performed in the accelerator facility at GSI Darmstadt, Germany. H-like $^{140}Pr^{59+}$ and $^{142}Pm^{60+}$ were produced by fragmentation of primary beams of ^{152}Sm produced at the heavy ion synchroton SIS. The primary ^{152}Sm beam typically has energies in the range of 500-600 MeV per nucleon, and the fragment separator FRS separates the primary beam into individual beams of $^{140}Pr^{59+}$ and $^{142}Pm^{60+}$ and feeds them into the ESR storage ring (See Physics Letters B 664 (2008) [4] Page 162). The ESR storage ring has the unique function of storing exotic nuclei with the purpose of measuring their masses and lifetimes using the Schottky Mass Spectrometry technique. In the SMS technique, the ions are electron-cooled and their mass-to-charge ratios are determined by measuring their revolution frequencies. Similarly, the β lifetimes of stored, unstable nuclei can also be determined (See Journal of Physics B: AMO Physics 36 (2003) [1] Pages 585-597).

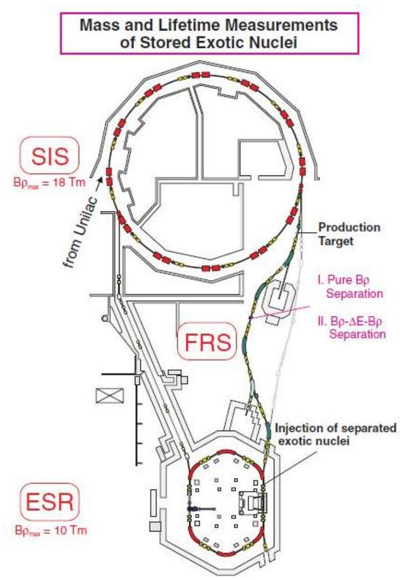

Figure 1: The GSI Facility.

1.2 The Decay Mechanisms

Both the ^{140}Pr and ^{142}Pm atoms decay either by electron capture (EC) or by emitting a positron (β^+). So it is a good idea to review these two types of decay mechanisms. In a β^+ decay, energy is used to convert a proton into a neutron, a positron and an electron neutrino:

$$energy + p \rightarrow n + \beta^{+} + \upsilon_e$$

Since a neutron is more massive than the proton, the binding energy of the mother nucleus has to be less than that of the daughter nucleus. The energy discrepancy between the daughter and mother nuclei is responsible for the conversion of a proton into a neutron, positron and an electron neutrino and the resulting kinetic energy of these particles (See Wikipedia: Beta Decay). An example of a β^{+} decay is given in Section 1. Processes that involve β^{+} decay are generally accompanied by an EC (electron capture) decay where the innermost electron interacts with the nuclei of an atom resulting in the emission of a neutrino (See Wikipedia: Beta Decay):

$$energy + p + e^{-} \rightarrow n + \upsilon_e$$

An example of an EC decay is as follows:

$${}^{22}_{11}Na + e^{-} \rightarrow^{22}_{10} Ne + \upsilon_e$$

In both the decays, the existence of electron neutrinos is unmistakable. A theory is that the oscillatory decay is a result of the superposition of mass eigenstates of the neutrino. In other words, neutrino oscillations or the change in neutrino flavor is the cause of the oscillatory term. A detailed decay scheme of ${}^{140}_{59}Pr$ and ${}^{142}_{61}Pm$ is given below:

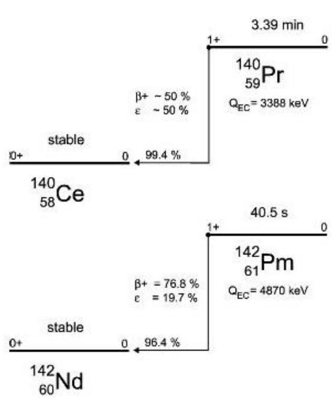

Figure 2: Decay Scheme of neutral ${}^{140}Pr$ and ${}^{142}Pm$ atoms.

As can be seen, both the nuclei decay to stable daughter nuclei either by the three-body positron emission or the two-body EC decay. The varying decay energies and lifetimes of the decays allow for the detailed comparison of the time evolution of the decays.

1.3 The GSI Data and Discussion

The decay times of the ions were measured after the ions were injected into the ESR for cooling. The times of appearance of the daughter nuclei were measured and the data were fitted with the exponential decay function:

$$\frac{dN_{EC}(t)}{dt} = N(0) * \lambda_{EC} * e^{-\lambda t}, \quad (1)$$

where $\frac{dN_{EC}(t)}{dt}$ is the rate of decay of the ions due to electron capture, $N(0)$ is the total number of the parent ions at $t = 0$, λ_{EC} is the electron-capture decay constant and λ is the total decay constant (not only due to electron capture). However, as can be seen from Figure 3 (See Physics Letters B 664 (2008) [4] Page 165), an exponential fit is clearly not sufficient to explain the superimposed periodic time modulation. One thing visible about equation

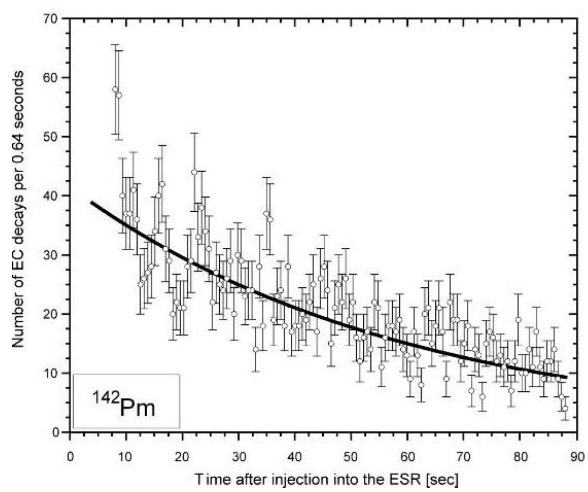

Figure 3: Rate of EC decays of ^{142}Pm after injection into the ESR ring.

(1) is that the decay constant λ is a constant and is not a function of time. In order to account for the periodic modulation, λ has to vary with time,

which will be shown later in the paper. So equation (1) now becomes:

$$\frac{dN_{EC}(t)}{dt} = N(0) * \tilde{\lambda_{EC}}(t) * e^{-\lambda t}. \tag{2}$$

And,

$$\tilde{\lambda_{EC}} = \lambda[1 + acos(\omega t + \phi)], \tag{3}$$

where λ is some constant, a is the amplitude of oscillation, ω is the angular frequency and ϕ is the initial phase of the modulation. Decays through electron capture were only considered as it was thought that the modulation was only due to electron capture. If the fitting is done using equation (2), the decay of ^{142}Pm looks as follows:

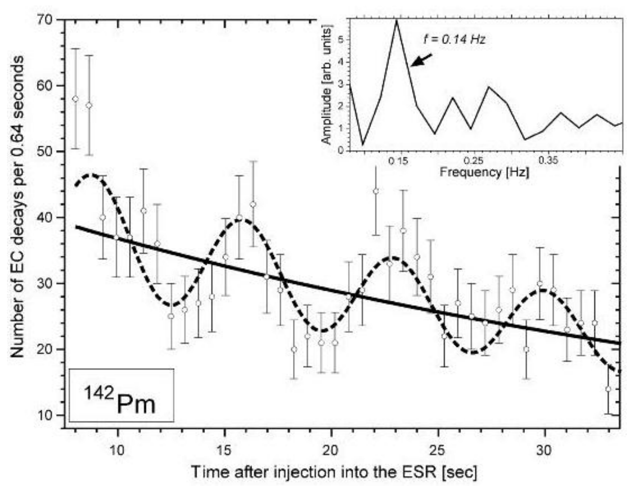

Figure 4: Solid line shows fit according to equation (1) and the dashed line for equation (2).

Now the question is how to account for the oscillatory term. The original team of investigators hypothesized that the difference in the mass eigenstates of the neutrinos, due to neutrino oscillations, can explain for the discrepancy. According to them, (See Physics Letters B 664 (2008) [4] Page 167) by the law of conservation of energy:

$$E_1 + M + \frac{p_1^2}{2M} = E \tag{4}$$

$$E_2 + M + \frac{p_2^2}{2M} = E, \tag{5}$$

where E is the energy of the initial state, $E_i = \sqrt{p_i^2 + m_i^2}$ represents the energy of the two neutrino mass eigenstates with masses m_1 and m_2 respectively, $p_i^2/2M$ represents the respective kinetic energies of the recoiling daughter nuclei and M is the mass of the daughter nucleus. By combining equations (4) and (5), we get the following:

$$\Delta E = E_2 - E_1 \approx \frac{\Delta m^2}{2M}, \tag{6}$$

where $\Delta m^2 = m_1^2 - m_2^2$. According to the GSI team, the energy splitting ΔE is responsible for the modulations. However, research teams from different parts of the globe do not seem to agree with the original hypothesis; one team in UC Berkeley even rejects the possibility of an oscillatory behavior through their experimental results with ^{142}Pm (See Physics Letters B 670 (2008) [3] Pages 196-199). This result should be taken with a grain of salt as the experiment was performed in a different setting using different instruments. On the other hand, some theoreticians predict that the oscillations of the orbital $K-$shell electron capture decay (EC) rate of the $H-$like heavy ions are caused by quantum beats of two coherently excited, closely spaced masseigenstates of decaying $H-$like heavy ions (See Research Letters in Physics Volume 2009 [2] Pages 1-4). In other words, the oscillations cannot be attributed to any form of neutrino mixing. This paper is just a basic approach to understand the mechanism behind the oscillations using very fundamental tools of quantum mechanics.

2 The Mechanism

Before plunging deeper into the oscillations, it would be worthwhile to find out what makes a radioactive decay exponential in the first place. This section would deal with the fundamentals of a nuclear decay starting from simple first principles.

2.1 Quantum Tunneling and Radioactive Decay

Quantum tunneling is a phenomenon that essentially involves a particle with energy E and a barrier with potential U_0, where $E < U_0$. The particle should not get through the barrier classically; however, there is a probability that it will. So even though classically the particle is in a 'bound state', quantum mechanically the particle is in a 'scattering state'. It was in 1928 that George Gamow incorporated the theory of quantum tunneling to explain alpha decay (See Introduction to Quantum Mechanics [6] Page 334). First of all, let's have a review of the potential barrier.

2.2 The Potential Barrier

Basically, there is a barrier that extends from $x = 0$ to $x = L$ such that:

$$\mathrm{U}(x) = \begin{cases} 0, & x < 0, x > L \\ U_0, & 0 < x < L \end{cases} \tag{7}$$

Let the regions $x < 0$, $0 < x < L$ and $x > L$ be denoted by I, II and

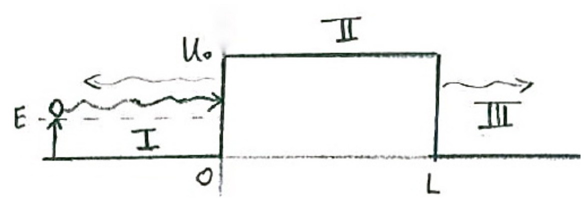

Figure 5: A Potential Barrier.

III and their corresponding wavefunctions be denoted by Ψ_I, Ψ_{II} and Ψ_{III} respectively. The wavefunctions can be written as follows:

$$\psi_1 = Ae^{ikx} + Be^{-ikx}, \tag{8}$$

$$\psi_2 = Ce^{lx} + De^{-lx}, \tag{9}$$

$$\psi_3 = Fe^{ikx}, \tag{10}$$

where $k = \sqrt{2mE}/\hbar$ and $l = \sqrt{-2m(E - U_0)}/\hbar$. k and l are the constants that came out of the time-independent Scrodinger's Equation. For example, in region II ($0 < x < L$),

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} + U_0\psi_{II} = E\psi_{II}.$$

Or,

$$\frac{d^2\psi_{II}}{dx^2} = \frac{-2m}{\hbar^2}(E - U_0)\psi_{II}.$$

Since $E < U_0$, let

$$l^2 = \frac{-2m}{\hbar^2}(E - U_0).$$

Therefore,

$$l = \frac{\sqrt{-2m(E - U_0)}}{\hbar}.$$

The same holds true for k, in which case, $V = 0$ in the Scrodinger's Equation. In the regions $x < 0$ and $x > L$, the particle acts as a free particle; A is the amplitude of the incoming wave, B is the amplitude of the wave reflected from the barrier and F is the amplitude of the transmitted wave. As the particle is coming from left to right, region III does not have a reflected component. On the other hand, in region II ($0 < x < L$), the wave behaves exponentially and predominantly follows a decay pattern. The boundary conditions of the problem dictates that at $x = 0$:

$$\psi_I(x = 0) = \psi_{II}(x = 0).$$

And,

$$\frac{d\psi_I}{dx}(x = 0) = \frac{d\psi_{II}}{dx}(x = 0).$$

Similarly, at $x = L$,

$$\psi_{II}(x = L) = \psi_{III}(x = L).$$

And,

$$\frac{d\psi_{II}}{dx}(x = L) = \frac{d\psi_{III}}{dx}(x = L).$$

The wavefunctions have to be continuous and equal to each other at the boundary points. Applying the above mentioned constraints give us four equations which are as follows:

$$A + B = C + D. \tag{11}$$

$$A - B = \frac{il}{k}(D - C). \tag{12}$$

$$Ce^{lL} + De^{-lL} = Fe^{ikL}. \tag{13}$$

$$Ce^{lL} - De^{-lL} = \frac{ik}{l}Fe^{ikL}. \tag{14}$$

By eliminating C and D, and solving for F in terms of A, we get:

$$F = \frac{2iklA}{e^{ikl}[2ilkcosh(lL) - sinh(lL)(l^2 - k^2)]}. \tag{15}$$

The transmittance or *transmission coefficient* can be written as (See Introduction to Quantum Mechanics [6] Page 87):

$$T \equiv \frac{|F|^2}{|A|^2}. \tag{16}$$

In our case, by plugging the expression for F in equation (16) and expressing the answer in terms of E and U_0, the transmission coefficient comes out to be:

$$T = \frac{4(\frac{E}{U_0})(1 - \frac{E}{U_0})}{sinh^2\left(\frac{\sqrt{2m(U_0-E)}}{\hbar}\right) + 4(\frac{E}{U_0})(1 - \frac{E}{U_0})}. \tag{17}$$

Moreover, under the "wide barrier" assumption, equation (17) simplifies further. In the barrier region (Region II), De^{-lx} dominates, or $De^{\frac{-x}{\delta}}$ where, $\delta = \frac{1}{l}$ has units of length (See Modern Physics [7] Page 202). If $L >> \delta$, very little of the wavefunction survives to $x = L$. Hence the condition for a "wide" barrier is:

$$\frac{L}{\delta} = lL = \frac{\sqrt{2m(U_0 - E)}L}{\hbar} >> 1. \tag{18}$$

Upon applying this condition to equation (17), it simplifies to:

$$T = 16\left(\frac{E}{U_0}\right)\left(1 - \frac{E}{U_0}\right)e^{\frac{-2L\sqrt{2m(U_0-E)}}{\hbar}}. \tag{19}$$

As one can see, there is an exponential term which is a direct consequence of writing the $sinh^2$ term in the denominator of equation (16) in terms of $sinh^2(z) = \left(\frac{e^z - e^{-z}}{2}\right)^2$ and applying condition (18).

2.3 Transmission Coefficient and Radioactive Decay

If we consider a radioactive decay, for example an alpha decay, and suppose we just take one atom of $Uranium - 238$ that converts to $Thorium - 234$

upon alpha decay (see Section 1), then the decay rate can be modelled as follows (See Modern Physics [7] Page 209):

$$\frac{N}{time} = \frac{N_\alpha}{time_d} * T,$$

where N =number of decays, N_α = number of times α strikes barrier, $time_d$ =time taken to cross the nuclear diameter and T =transmission probability. It can be further written that:

$$\frac{N}{time} = \frac{1 - strike}{time_d} * T.$$

Or,

$$\frac{N}{time} = \frac{v}{2r_{nuc}} * T,$$

where v =velocity of the α particle and $2r_{nuc}$ = diameter of the nucleus. One thing that we have to keep in mind is that the calculation was for only one Uranium atom. However, for an aggregate of Uranium atoms, the equation changes to:

$$\frac{dN}{dt} = \frac{v}{2r_{nuc}} * T * N(t).$$

Or,

$$\frac{dN(t)}{dt} = \lambda * N(t), \tag{20}$$

which is a first-order, linear differential equation with constant coefficients and a solution:

$$N(t) = N(0)e^{-\lambda t}. \tag{21}$$

And,

$$\lambda = \frac{v}{2r_{nuc}} * T. \tag{22}$$

The coefficient, λ, can either be constant or a function of time as evident in the GSI case. In that case, the solution, $N(t)$, can be significantly different as will be shown later. How can λ vary with time? By just looking at equation (22), one can just infer that it has something to do with T. The next few sections would deal with how T can become a function of time, which, in turn, makes λ vary with time.

2.4 A Slightly Different Case

Let's look at a barrier, again extending from $x = 0$ to $x = L$, such that:

$$\mathrm{U}(x) = \begin{cases} V_0, & x < 0 \\ V_1, & 0 < x < L \\ 0, & x > 0 \end{cases} \tag{23}$$

In this case, the wavefunctions Ψ_I, Ψ_{II} and Ψ_{III} would look the same as

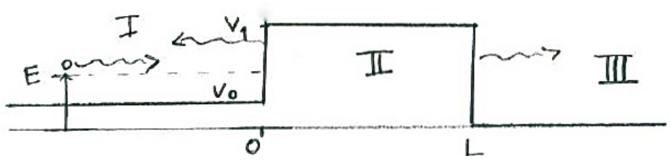

Figure 6: A slightly modified Potential Barrier.

before but with a different constant in the exponential term of Ψ_I, as region I ($x < 0$) now has a potential of V_0. Hence the wavfunctions are:

$$\psi_1 = Ae^{ikx} + Be^{-ikx}, \tag{24}$$

$$\psi_2 = Ce^{lx} + De^{-lx}, \tag{25}$$

$$\psi_3 = Fe^{imx}, \tag{26}$$

where $k = \sqrt{2m(E - V_0)}/\hbar$, $l = \sqrt{-2m(E - V_1)}/\hbar$ and $m = \sqrt{2mE}/\hbar$. If we turn the krank and do as before, T would be:

$$T = \frac{\frac{4\left(1-\frac{E}{V_1}\right)(E-V_0)}{V_1\left(1-\frac{V_0}{V_1}\right)}}{sinh^2\left(\frac{a\sqrt{2m(V_1-E)}}{\hbar}\right) + \frac{\left(1-\frac{E}{V_1}\right)(2E-V_0)}{V_1\left(1-\frac{V_0}{V_1}\right)} + \frac{2\left(1-\frac{E}{V_1}\right)\sqrt{E}\sqrt{E-V_0}}{\left(1-\frac{V_0}{V_1}\right)V_1}}. \tag{27}$$

Equation (27) can be checked by taking $V_0 = 0$ and it would simplify to equation (17), which we derived earlier. It is quite amazing to see how just adding one parameter in the problem, can make the expression of T so much more sophisticated. Similarly, using the "wide barrier" approximation, as in Section 2.2, this new T would become:

$$T = \frac{16\left(1-\frac{E}{V_1}\right)(E-V_0)}{V_1\left(1-\frac{V_0}{V_1}\right)} e^{\frac{-2L\sqrt{2m(V_1-E)}}{\hbar}} \tag{28}$$

2.5 Deriving a Time-Dependent $\lambda(t)$

Building up on Section 2.4, another parameter can be introduced into the problem. What if $V_0(t) = V cos(\omega t)$, oscillating as a function of time with some amplitude V and a frequency $f = \omega/2\pi$? In this case,

$$\mathrm{U}(x) = \begin{cases} V cos(\omega t), & x < 0 \\ V_1, & 0 < x < L \\ 0, & x > 0 \end{cases} \tag{29}$$

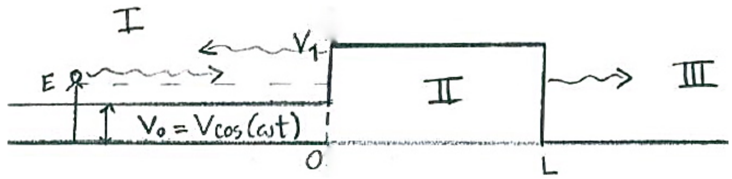

Figure 7: Potential Barrier with an alternating V_0.

Equation (28) now changes to:

$$T = \frac{16\left(1 - \frac{E}{V_1}\right)(E - V cos(\omega t))}{V_1\left(1 - \frac{V cos(\omega t)}{V_1}\right)} e^{\frac{-2L\sqrt{2m(V_1 - E)}}{\hbar}}. \tag{30}$$

Using equation (22) for λ it can be written that:

$$\lambda(t) = \frac{8v}{r} e^{\frac{-2L\sqrt{2m(V_1 - E)}}{\hbar}} \frac{\left(1 - \frac{E}{V_1}\right)(E - V cos(\omega t)}{V_1\left(1 - \frac{V cos(\omega t)}{V_1}\right)}. \tag{31}$$

And let,

$$K = \frac{8v}{r} e^{\frac{-2L\sqrt{2m(V_1 - E)}}{\hbar}} \left(1 - \frac{E}{V_1}\right).$$

Therefore $\lambda(t)$ becomes:

$$\lambda(t) = \frac{K(E - V cos(\omega t))}{V_1 - V cos(\omega t)}. \tag{32}$$

Let $V << V_1$ and implementing binomial expansion on the denominator of equation (32), the final expression of $\lambda(t)$ comes out to be:

$$\lambda(t) = \frac{EK}{V_1}\left[1 + \left(\frac{V}{V_1} - \frac{V}{E}\right) cos(\omega t + \tilde{\phi})\right], \tag{33}$$

which has an unmistakably close resemblance to $\tilde{\lambda}(t)$ as shown earlier in equation (3).

3 Comparing Models

Now that we have shown that λ varies as a function of time using our own model, it would be nice to compare our model with the one given by the GSI team. Essentially, the two models can be described by equation (2) and our own assumption:

$$\frac{dN(t)}{dt} = N(t) * \lambda(t), \tag{34}$$

where $\lambda(t)$ is the one given by equation (33) and $N(t)$ is the number of ions at any given time t. Hopefully, evaluating the functions $N_{EC}(t)$ and $N(t)$ and plotting them would give us some insight about what is really going on.

3.1 Solving $N_{EC}(t)$ using the GSI Model

After plugging the expression for $\tilde{\lambda_{EC}}(t)$ in equation (2), it turns out to be:

$$\frac{dN_{EC}(t)}{dt} = N(0) * e^{-\lambda t} * \lambda_{EC}(1 + acos(\omega t + \phi)).$$

Or,

$$dN_{EC}(t) = N(0) * \lambda_{EC} * e^{-\lambda t} * [1 + acos(\omega t + \phi)]\, dt.$$

Or,

$$\int_{n_{EC}}^{N_{EC}} dN_{EC}^{'} = N(0) * \lambda_{EC}\left[\int_0^t e^{-\lambda t^{'}} dt^{'} + a\int_0^t e^{-}\lambda t^{'} cos(\omega t^{'} + \phi)dt^{'}\right].$$

Finally, upon solving,

$$N_{EC}(t) = n_{EC} + N(0) * \lambda_{EC}\left[\frac{1}{\lambda} - \frac{e^{-\lambda t}}{\lambda} + \frac{a}{\omega^2 + \lambda^2}\left(\lambda cos(\phi) - \omega sin(\phi) + e^{-\lambda t}\omega sin(\omega t + \phi) - \right.\right.$$

$$\lambda e^{-\lambda t} cos(\omega t + \phi).] \quad (35)$$

This function can be plotted for the ion ^{142}Pm using the fitting parameters given in the paper (See Physics Letters B 664 (2008) [4] Page 166). The graph looks as follows:

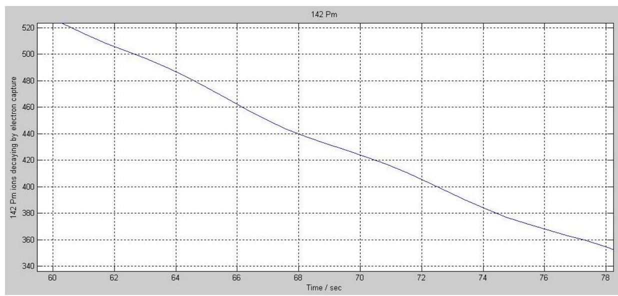

Figure 8: Number of EC decays of ^{142}Pm.

Figure 8 shows a magnified portion of the decay curve. It is important to mention that the graphs in Figures 3 and 8 represent quite different phenomena. Figure 3 represents the rate of decay of ^{142}Pm with time; on the other hand, Figure 8 represents the number of ^{142}Pm ions that are actually present at any given time. As expected, the number of ions decrease in an exponential fashion. Only if the graph is magnified one can discern the small 'bumps' riding on top of the exponential decay. The period of these bumps is around 7s as shown by the experiment. The amplitude of the 'bumps' depends on a, which is only 0.23 according to the fitting parameters (See Physics Letters B 664 (2008) [4] Page 166). The next step would be to see whether our model corresponds to something similar to the GSI model.

3.2 Solving $N(t)$ using our Model

Our model can also be dealt with similarly by plugging the expression for $\lambda(t)$, equation (33), into equation (34). By making some small changes in variables, $\lambda(t)$ can be written as:

$$\lambda(t) = H[1 + Mcos(\omega t + \tilde{\phi})], \quad (36)$$

where $H = \frac{EK}{V_1}$ and $M = \left(\frac{V}{V_1} - \frac{V}{E}\right)$, and equation (34) would look like:

$$\frac{dN(t)}{dt} = N(t) * H[1 + Mcos(\omega t + \tilde{\phi})]. \quad (37)$$

After solving equation (37), the solution turns out to be a rather simple expression for $N(t)$ as follows:

$$N(t) = \tilde{N(0)} e^{-H\left[t + \frac{M}{\omega} sin(\omega t + \tilde{\phi})\right]}. \tag{38}$$

Upon plotting $N(t)$ for some arbitrary values of M and keeping the ω constant, the following could be seen:

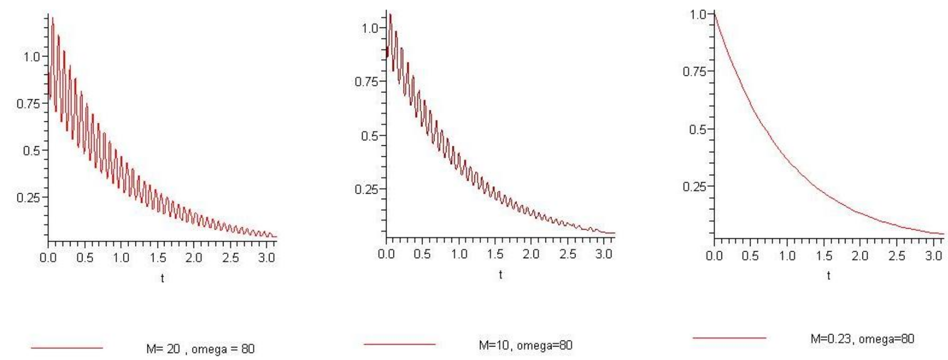

Figure 9: The effect of changing M on $N(t)$.

As can be seen from Figure 9, the variable M acts as the amplitude of the periodic function, which is accounted for by the exponent of the exponential term of $N(t)$. As M gets smaller and smaller, N(t) gets almost exponential and the oscillations merge with the exponential function itself. What we have seen in Figure 8 were very small 'bumps' accounted for by the very small amplitude of the oscillatory term as well. Therefore, in essence, both $N(t)$ and $N_{EC}(t)$ look similar in the limit when $M \to 0$. The next step is to find out what each term in $\frac{dN(t)}{dt}$ corresponds to each term in $\frac{dN_{EC}(t)}{dt}$.

3.3 Comparing $\frac{dN(t)}{dt}$ and $\frac{dN_{EC}(t)}{dt}$

Now that $N_{EC}(t)$ and $N(t)$ are known, $\frac{dN_{EC}(t)}{dt}$ and $\frac{dN(t)}{dt}$ can be finally written as:

$$\frac{dN_{EC}(t)}{dt} = N(0) * \lambda_{EC} * e^{-\lambda t} * (1 + acos(\omega t + \phi)). \tag{39}$$

And,

$$\frac{dN(t)}{dt} = \tilde{N(0)} * H * e^{-H\left[t + \frac{M}{\omega} sin(\omega t + \tilde{\phi})\right]} * \left(1 + Mcos(\omega t + \tilde{\phi})\right). \tag{40}$$

We can think of equation (40) as a function of both M and t, and Taylor expand it around $M = 0$ keeping t constant. This is relevant because it is important to see the way $\frac{dN(t)}{dt}$ behaves for values of M close to zero and whether it is a good approximation for $\frac{dN_{EC}(t)}{dt}$. After the expansion around $M = 0$, $\frac{dN(t)}{dt}$ looks as follows:

$$\frac{dN(t)}{dt} = \tilde{N(0)} * H * e^{-Ht} * \left(1 - \frac{MH}{\omega} sin(\omega t + \tilde{\phi})\right) \qquad (41)$$

This looks awefully familiar to the expression for $\frac{dN_{EC}(t)}{dt}$ and convenient enough for the comparison of the different terms between the two. Upon comparing the different terms, it is pretty obvious that $\tilde{\phi} = \phi + \frac{\pi}{2}$, $\frac{MH}{\omega} = -a$, $H = \lambda$, $\omega = \omega$, and $\tilde{N(0)}H = N(0)\lambda_{EC}$.

3.4 Some Calculations

Using the case of ^{142}Pm, we have the following:

$$\tilde{\phi} = \phi + \frac{\pi}{2} = -2.92 \times 10^{-2} rad.$$

$$\omega = 0.885 s^{-1}.$$

$$\lambda = H = \frac{EK}{V_1} = 0.0224 s^{-1}.$$

$$M = \frac{-a\omega}{H} = \left(\frac{V}{V_1} - \frac{V}{E}\right) = -9.09.$$

The values of λ, ω and ϕ were taken straight from the GSI paper (See Physics Letters B 664 (2008) [4] Page 166). Just a refresher - V represents the amplitude of oscillation of $V_0(t)$, V_1 represents the height of the potential barrier (See Section 2.5) and E represents the energy with which the particle is approaching the barrier. In other words, V_1 stands for the potential that a particle has to overcome in order to get out of the nucleus. It would be interesting to see what the ratios $\frac{V_1}{E}$ and $\frac{V}{E}$ turn out to be - just another way to check whether our model makes any sense.

Even though EC decays are the ones of importance, it would be quite hard to find numerical values of the different parameters of our model using only the EC decays. This is because the only other decay product, with the exception of the daughter nucleus, of an EC decay is a neutrino, which has negligible mass. So applying the law of conservation of momentum, would hardly give us any information regarding the different parameters of our model such as V, V_1 and so forth. As mentioned earlier, EC decays are also accompanied by positron decays. In this case, the positrons would be ideal as they have the same mass as electrons and applying conservation of momentum is convenient enough to prove the feasibility of our model.

The end products of a positron decay of ^{142}Pm are a positron, the daughter nucleus ^{142}Nd and an electron neutrino. After applying the law of conservation of momentum on a single positron decay of ^{142}Pm, it is found that the velocity, v_{β^+}, with which a positron is ejected from the nucleus happens to be $2.98 \times 10^-8\ ms^{-1}$, which is very close to the speed of light. The recoil energy of the daughter nucleus ^{142}Nd is 90eV (See Physics Letters B 664 (2008) [4] Page 164), and was used in calculating the positron velocity. The velocity with which the positron is ejected from the nucleus is the same as the one with which it was approaching the potential barrier. The relativistic kinetic energy, E, of the positron is $6.29 \times 10^{-13} J$ or 3.93MeV and plugging this back in $\frac{EK}{V_1} = 0.0224s^{-1}$, the height of the potential barrier V_1 comes out to be $3 \times 10^{-6} J$. Now that V_1 and E are known, the amplitude of oscillation, V, of $V_0(t)$ can be determined and is equal to $5.72 \times 10^{-12} J$. Consequently,

$$\frac{V_1}{E} = 4.77 \times 10^6.$$

And,

$$\frac{V_1}{V} = 5.24 \times 10^5.$$

The numbers are in agreement with the expectations. It can be seen that the height of the potential barrier is several orders of magnitude larger than both the energy of the positron and the amplitude of oscillations of V_0. Finally, the question is what brings about the oscillations of V_0.

4 A Possible Explanation

One possible explanation of the oscillations is given by the 'quantum beats' theory derived from the idea of closely spaced energy eigenstates of the decaying mother ions (See Nuclear Physics B 188 (2009) [5] Pages 43-45). So far this is the only theory that has not been proven totally wrong. Can we explain this theory in the light of something analogous that we have already come across? The oscillations in our model for $V_0(t)$, cannot be literal oscillations of the potential itself. There has to be some inner mechanism that is manifesting itself in the form of these oscillations. A possible cause of such oscillations might be due to the interaction of the external magnetic field (earth and/or the particle accelerator) with the nuclear spin of the mother ions.This section would be dealing with building an analogy between an electron in a magnetic field and our model.

4.1 The Analogy

An electron is a particle with an intrinsic spin, and has a magnetic dipole moment, μ proportional to the spin angular momentum, S (See Introduction to Quantum Mechanics [6] Page 190):

$$\mu = \gamma S. \tag{42}$$

An external magnetic field (e.g $B = B_0\hat{k}$) exerts a torque $\mu \times B$ on the magnetic dipole moment and the energy associated with that change is given by the Hamiltonian $H = -\mu \cdot B$ or $H = -\gamma B \cdot S$. Since the magnetic field is acting along the z axis, only the z components come into play and the energy becomes:

$$H = -\gamma B_0 S_z = \frac{-\gamma B_0 h}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is known that electrons can be either 'spin-up' or 'spin-down' represented by the spinors χ_+ and χ_- with the corresponding energies of $E_+ = \frac{-\gamma B_0 h}{2}$ and $E_- = \frac{\gamma B_0 h}{2}$. Hence the spin state of the electrons can be expressed as a linear combination of the two states given by:

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar}. \tag{43}$$

For convenience, let $a = cos(\frac{\alpha}{2})$ and $b = sin(\frac{\alpha}{2})$ (See Introduction to Quantum Mechanics [6] Page 192). Therefore,

$$\chi(t) = cos\left(\frac{\alpha}{2}\right)\chi_+ e^{-iE_+t/\hbar} + sin\left(\frac{\alpha}{2}\right)\chi_- e^{-iE_-t/\hbar}. \tag{44}$$

The spin S has components S_x, S_y and S_z. The expectation values of these quantities are as follows:

$$\langle S_x \rangle = \chi(t)^\dagger S_x \chi(t) = \frac{\hbar}{2} sin(\alpha) cos(\gamma B_0 t)$$

$$\langle S_y\rangle = \chi(t)^\dagger S_y \chi(t) = \frac{-\hbar}{2} sin(\alpha) sin(\gamma B_0 t)$$

$$\langle S_z\rangle = \chi(t)^\dagger S_z \chi(t) = \frac{\hbar}{2} cos(\alpha)$$

It is not very hard to see that the components S_x and S_y are oscillating with some frequency γB_0 making the spin S precess along the z axis with some angle α. The frequency of precession is known as *Larmor Precession Frequency* given by:

$$\omega = \gamma B_0. \tag{45}$$

In the classical sense, the spin is 'precessing'; however, quantum mechanically the precession is associated with the different energies inherent within the two different spin states. The *Larmor Precession Frequency* can also be written in terms of energy as:

$$E_{Larmor} = \hbar\omega = \hbar\gamma B_0. \tag{46}$$

It can be easily seen that it is exactly equal to the energy difference of the two different spin-states of the electron.

4.2 The Analogy put into Practice

The electron spin analogy can be used to understand the oscillations of the GSI anomaly. Many of the nuclei have no overall spin; however, some nuclei do have an overall spin. The rules for determining spin are as follows (See the webpage *Theoretical Physics*, Sheffield Hallam University):

1. If the number of neutrons and the number of protons are both even, then the nucleus has NO spin.

2. If the number of neutrons plus the number of protons is odd, then the nucleus has a half-integer spin (i.e. 1/2, 3/2, 5/2).

3. If the number of neutrons and the number of protons are both odd, then the nucleus has an integer spin (i.e. 1, 2, 3).

$^{142}Pm^{60+}$ has 61 protons and 81 neutrons; similarly, $^{140}Pr^{58+}$ has 59 protons and 81 protons. This suggests that both the ions have integer spin and is susceptible to the effects of external magnetic field. Quantum mechanics tells us that a nucleus of spin I will have $2I + 1$ possible orientations. A nucleus with spin 1/2 will have 2 possible orientations. In the absence of an

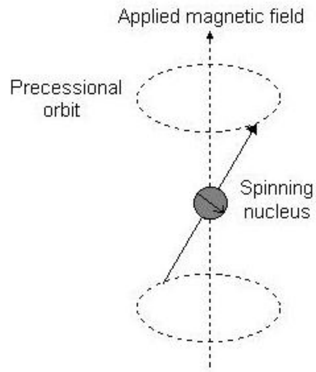

Figure 10: Precession of Nuclear Spin in a Magnetic Field.

external magnetic field, these orientations are of equal energy. If a magnetic field is applied, then the energy levels split. Each level is given a magnetic quantum number, m. Just like electrons, the ions in the GSI experiment exist in several different energy states so that the overall state is a superposition of the different spin eigenstates with the same decay rate γ (See Nuclear Physics B 188 (2009) [5] Pages 45):

$$I(t) = (A_1 e^{-iE_1 t} I_1 + A_2 e^{-iE_2 t} I_2) e^{-\gamma t/2} \tag{47}$$

Likewise, it can be said that nuclear spins precess and their precession is a consequence of the varying energies associated with the various energy eigenstates of the mother nuclei.

5 Conclusion

We have successfully come up with a model that produces the fitting equations of the GSI model and also formulated an explanation showing why the 'quantum beats' theory might work. However, a lot needs to be done in terms of calculating the energy split between the eigenstates and finding out the exact source of the external magnetic field. We are not saying that our model is the absolute way of looking at things, but getting an agreement with the GSI team shows that it is a model to be looked at for future work.

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